Mathematics IB Tutorial 7 (week 8)

Semester 2, 2016

1. (a) (i) Let V be a subspace of $\mathbb{R}^n$ with orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$. Denote by $\mathrm{pr}_V : \mathbb{R}^n \to \mathbb{R}^n$ the linear transformation given by orthogonal projection onto V. Prove that

$$\mathrm{pr}_V(\mathbf{x}) = (\mathbf{v}_1\mathbf{v}_1^t + \cdots + \mathbf{v}_r\mathbf{v}_r^t)\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

(ii) Let A be a symmetric $n \times n$ matrix and $P = (\mathbf{u}_1 \cdots \mathbf{u}_n)$ be a matrix that orthogonally diagonalises A, so that $P^tAP = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Prove that

$$A = \lambda_1\mathbf{u}_1\mathbf{u}_1^t + \cdots + \lambda_n\mathbf{u}_n\mathbf{u}_n^t.$$

(b) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$.

(i) The eigenvectors of A are $(1, -2)$ and $(2, 1)$. Find the eigenvalue of each of these.

(ii) Check the formula in part (a) (ii) for this particular matrix.

(iii) Use this example, together with part (a) (i), to interpret the formula in part (a) (ii) geometrically. What is it saying?

2. A pentagon with a perimeter of 90cm is to be constructed by adjoining an equilateral triangle to a rectangle. Find the dimensions of the rectangle and triangle that will maximise the area of the pentagon.