1. Show that two of the following operators are linear and that one is not.

$$A(f(x)) = \frac{\partial f}{\partial x} + 3f(x) \qquad B\big(f(x)\big) = \frac{1}{2}f(x)\frac{\partial f}{\partial x} \qquad C(f(x)) = \int_0^5 (x-y)^3 f(y)dy$$

2. Consider the inner product space consisting of all linear combinations of $\sin(x)$ and $\cos(x)$ with the inner product:

$$\langle f|g\rangle = \int_0^{\pi} f^*(x)g(x)dx$$

a) What is the matrix representation of the operator $i\frac{\partial}{\partial x}$ if we take $\sin(x)$ and $\cos(x)$ to be our basis functions?

b) Show that the vectors $|p\rangle = e^{ix}$ and $|q\rangle = e^{-ix}$ are orthogonal in this space.

c) Find the scalar a so that $a|p\rangle$ and $a|q\rangle$ form an orthonormal basis.

d) What is the matrix representation of $i\frac{\partial}{\partial x}$ in this basis?

e) What is special about the basis formed by $a|p\rangle$ and $a|q\rangle$?

3. Using the inner product $\langle f|g\rangle = \int_a^b f^*(x)g(x)dx$ in a vector space for which all vectors satisfy $f(a) = f(b) = 0$, show that $i\frac{\partial}{\partial x}$ is a Hermitian operator but that $\frac{\partial}{\partial x}$ is not.

4. Consider the vector space of all 20 by 20 complex-valued matrices. Show that one of the following forms involving the matrix trace (Tr) is a valid inner product but that the other is not. Here we use A^+ to represent the conjugate transpose of the matrix A. (Hint: singular value decompositions are very useful things, and matrix traces and unitary matrices have many interesting properties.)

$$\langle A|B\rangle = \mathrm{Tr}(AB) \qquad \langle A|B\rangle = \mathrm{Tr}(A^+B)$$