Directions: Please write up your solutions and submit them on GauchoSpace by the end of the day on Monday. We prefer the submission to be single pdf file if possible. Bonus points will be awarded for organization and neatness.

Lesson 11 Variance

1. We have an experiment that produces a measurement X with probability density function

x	-10	-5	0	5	10
$P(x)$	0.1	0.2	0.3	0.2	0.2

   (a) Calculate the standard deviation of X.

   (b) If we have 100 independent replications of this experiment with measurements X_i and the average of these 100 random variables is $\bar{X}$, then calculate $\mathbb{V}\text{ar}(\bar{X})$.

   (c) Use Chebyshev's inequality to bound the probability

$$\mathbb{P}\{\left|\bar{X} - 1\right| > 2\}$$

2. X is a binomial observation from 125 independent trials with probability $p = 0.3$, and Y is an independent binomial observation from 235 trials with the same probability $p = 0.3$. Calculate the standard deviation of

   (a) X

   (b) $X/125$

   (c) $\dfrac{X}{125} - \dfrac{Y}{235}$

Lesson 12 Infinite Sample Spaces

3. Suppose that each day there is a 10% chance of rain and whether or not it rains each day is independent of all the other days.

   (a) Calculate the probability that we will have to wait at least 5 days for it to rain for the first time (i.e. at least 4 consecutive non-rainy days before the first rain; i.e., it rains no earlier than the 5th day)

   (b) Calculate the expected number of non-rainy days before we get the first day that it rains.

4. The number of gold finch nests in Upper Arboreal environments is theorized to average about 8.76 per acre. We will suppose that these nests are placed according to a Poisson distribution.

   (a) What is the probability that there will be 7 or 8 nests in an acre?

   (b) Let Y be the number of nests counted in 15 acres. What is the expected value of Y?

   (c) What is the probability that $115 < Y < 120$?

Questions

5. An urn contains four chips numbered 2, 4, 6, and 10. Two are drawn without replacement. Let the random variable X denote the larger of the two numbers on the chips drawn. Calculate $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$.

6. Let W be a random variable that is equally likely to be any integer from 1 to 10.

(a) Calculate the $\mathbb{E}(W)$.

(b) Calculate the $\mathbb{V}\text{ar}(W)$.

(c) Calculate the $\mathbb{V}\text{ar}(10W)$.

(d) Calculate the variance of a random variable that is equally likely to be any integer from 1 to 100.

7. Suppose that X has a Geometric Distribution where

$$P(x) = p^{x-1}(1-p) \qquad \text{for } x = 1, 2, 3, 4, 5, \ldots$$

(a) Show that the expected value

$$\mathbb{E}(X-1)(X-2) = \frac{2p^2}{(1-p)^2}$$

by using the fact that

$$\sum_{x=1}^{\infty} P(x) = 1, \quad \sum_{x=1}^{\infty} \frac{d}{dp} P(x) = 0 \text{ and } \qquad \sum_{x=1}^{\infty} \frac{d^2}{dp^2} P(x) = 0.$$

Hint: $E(X-1) = \sum_{x=1}^{\infty}(x-1)P(x) = \sum_{x=1}^{\infty}(x-1)p^{x-1}(1-p)$

(b) Use the answer from part (a) to calculate the variance of X.

(c) Calculate $\mathbb{E}(a^X)$ for some number a that is $|a| < 1$.

(d) Use the answer from part (c) to calculate $\mathbb{E}(e^{tX})$ for some number t. For which values of t does this expectation not exist?

Group Question

Please submit your answer online. The Group that provides the simplest and clearest explanation for the correct answers will receive extra credit.

8. (a) Alice and Bob are playing a game where they each write their name on a card. Alice goes first and chooses one of the two cards with equal probability. If Alice chooses her own card then she wins. If she chooses Bob's card, then it is Bob's turn to play. He chooses one of the two cards randomly with equal probability, and he wins if gets his own card. Otherwise, the turn passes back to Alice and she tries again. This pattern continues until one or the other has won. Calculate the probability that Alice will win this game. (Hint: it is not 1/2.)

(b) Claire walks up and wants to join the game so she adds a card with her name into the pile. Now, if Alice draws Bob's card then he goes next and if she draws Claire's card then Claire goes next. At each turn, the person drawing a card will win if they draw their own card, and otherwise the turn will pass to whoever is named on the card. Every card is drawn independently with all three cards having equal probability. They continue to take turns until somebody wins. Calculate the probability of Alice winning, the probability of Bob winning, and the probability of Claire winning.