Problem Set 15

1. Let $Z_i \subseteq \mathbb{R}^N$, $i = 1, \ldots, n$ be a collection of zero measure sets. Show that the union $\bigcup Z_i$ also has zero measure.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded, integrable function.

   (a) Show that the graph of f, $\Gamma(f) := \{(x, f(x)) : x \in [a, b]\} \subseteq \mathbb{R}^2$ has zero content.

   (b) If f is non-negative, show that $S = \{(x, y) : a \leq x \leq b, 0 \leq y \leq f(x)\}$ is measurable, and $m(S) = \int_a^b f(x)dx$.

3. Show that if $f : \mathbb{R} \to \mathbb{R}^2$ is a C^1 function, then for any interval $I \subseteq \mathbb{R}$, $f(I)$ has zero Jordan measure.

4. If $S = \{x_1, \ldots, x_n\}$ is a finite set consisting of precisely n-elements, show that S has zero Jordan measure.

5. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. If $g : [a, b] \to \mathbb{R}$ is another function and $S = \{x : f(x) \neq g(x)\}$ contains exactly n-points, show that g is also Riemann integrable. [Note: You must prove this from scratch. If you wish to invoke a corollary or result from class, you must first prove it.]