Problem 1. *(4 points)* Let $S \subset \mathbb{R}$ be any set, and define for any $x \in \mathbb{R}$ the distance between x and the set S by

$$d(x, S) = \inf\{|x - s| \, : \, x \in S\}.$$

Prove the following:

(a) If $x \notin \bar{S}$ then $d(x, S) > 0$.

(b) The function $d_S : \mathbb{R} \to [0, +\infty)$, $d_S(x) = d(x, S)$, is Lipschitz continuous.

(c) If S is compact then for every $x \in \mathbb{R}$ there is $s \in S$ such that

$$|x - s| = d(x, S)$$

Problem 2. *(3 points)* If $f : E \to \mathbb{R}$ is a continuous function defined on some set $E \subset \mathbb{R}$ and if $\{a_j\}_{j \in \mathbb{N}} \subset E$ is a Cauchy sequence is it true that $\{f(a_j)\}_{j \in \mathbb{N}}$ is also a Cauchy sequence ? If true prove it. Otherwise provide a counterexample.

Problem 3. *(4 points)* Assume $f : [0, 1] \to [0, 1]$ is a continuous function. Show that there is $x \in [0, 1]$ with $f(x) = x$. Provide an example of a continuous function $g : (0, 1) \to (0, 1)$ where this is not true.

Problem 4. *(4 points)* Let $f : (0, +\infty) \to \mathbb{R}$ be a function which is twice differentiable and for which $f''(x) \geq c$ for all $x > 0$ and some constant $c > 0$. Prove that f is not bounded from above.