MATH 140
LIMITED LOGISTIC GROWTH MODEL ACTIVITY

Objective: Extend your understanding of the mathematical model for *limited logistic growth*, by generating graphs, describing model behavior for an application of the model with different scenarios of constrained growth and answering some analytical questions.

Reference: Chapter 4, section 4.5 of Blitzer's Textbook (Page 498)

General Instructions:

1. Read the information and complete the tasks outlined below. ***In this activity, there are a total of 8 questions to answer and 6 different graphs to generate (2 sets of axes, 3 graphs on each).***
2. Complete your assignment on the computer (*not* by hand), and submit your answers to the questions and graphs in either a MS Word or PDF file.
3. Complete this assignment *individually*. You may receive guidance from your classmates and the instructor via the instructor's **online office or student lounge.**
4. Answers to questions must be *in your own words*, using proper spelling and grammar. Graphs presented must be *generated by you* using the graphing tool designated within the activity. Appropriately cite any outside sources of information used such as:
   - "I copied some work from my tutor"
   - "Three of us divided up the work equally, so two-thirds of my submission is actually the work of my study buddies"
   - "Copied from web resources"

   Notes:

   (a) There will be an academic penalty depending on the extent of the unauthorized help, but you will not be held guilty of plagiarism *if* you cite them properly. However, if you use outside sources *without* citing them, you will be charged with plagiarism.

   (b) If a charge of plagiarism is proved, the consequence can be **severe**. For details, see the ERAU Plagiarism Tutorial. Plagiarized submissions, *at the minimum*, will receive a grade of zero.

5. ***This assignment is due in week 6 / module 6***, by the due date and time indicated in the course syllabus. To submit your completed assignment, go to: **Module 6 – Problem Set: Logistics Function Submission.** Click on the **Submit Assignment** button and follow the guidance to upload your file.

BACKGROUND

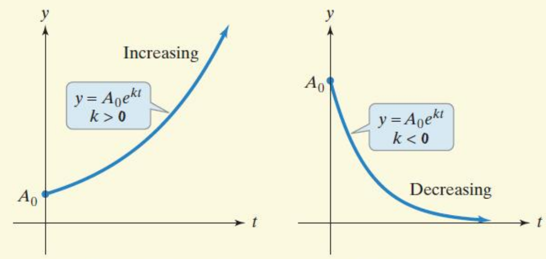
Exponential Growth and Decay

Our textbook, in section 4.5, describes the mathematical exponential growth and decay model:

The mathematical model for **exponential growth** or **decay** is given by

$$f(t) = A_0 e^{kt} \quad \text{or} \quad A = A_0 e^{kt}.$$

- **If $k > 0$, the function models the amount, or size, of a *growing* entity.** A_0 is the original amount, or size, of the growing entity at time $t = 0$, A is the amount at time t, and k is a constant representing the growth rate.
- **If $k < 0$, the function models the amount, or size, of a *decaying* entity.** A_0 is the original amount, or size, of the decaying entity at time $t = 0$, A is the amount at time t, and k is a constant representing the decay rate.



(a) Exponential growth **(b)** Exponential decay

In the real-world, pure exponential growth *can* take place for a short period of time, but *not* over an extended period of time. For example, the growth of any living species can start out as exponential growth, but over time, the growth is limited by the availability of space, resources, and other factors needed to sustain its growth. These limitations cause an eventual "leveling off" of the population.

Limited Logistic growth

Constrained exponential growth is known as *limited logistic growth*. As shown in section 4.5 of the textbook:

The mathematical model for limited logistic growth is given by

$$f(t) = \frac{c}{1 + ae^{-bt}} \quad \text{or} \quad A = \frac{c}{1 + ae^{-bt}},$$

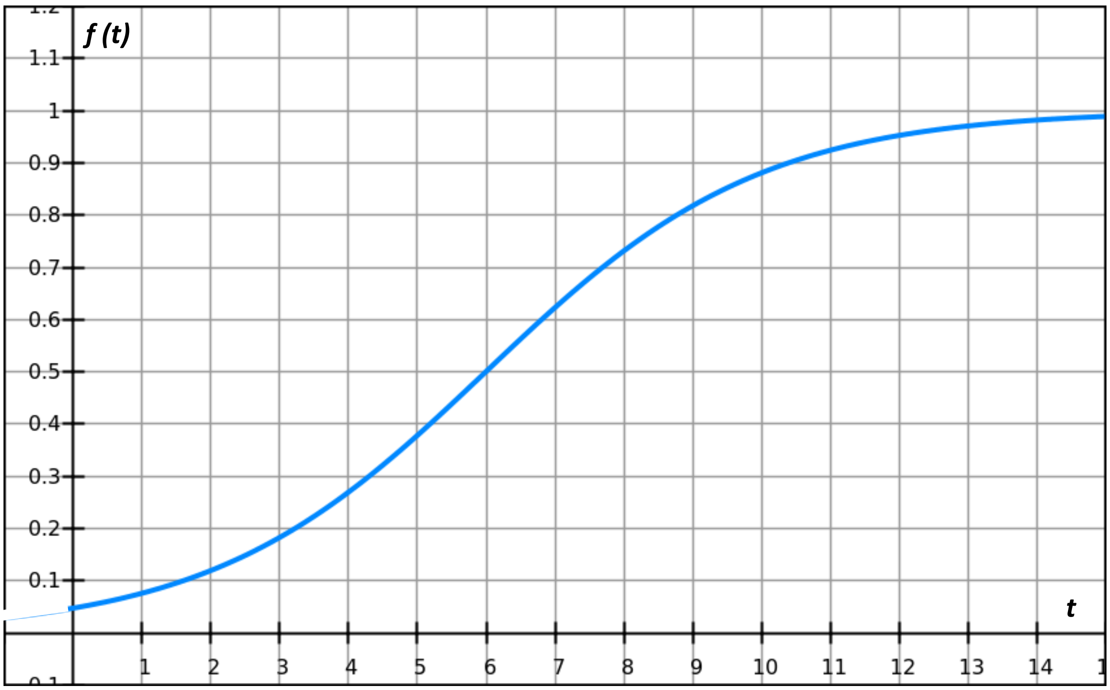
where a, b, and c are constants, with $c > 0$ and $b > 0$.

Activity

Part 1 - Characteristics of the Limited Logistic Growth Function

In the first part of the activity, you will analyze an equation and graph of a generic model of limited logistic growth, **f (*t*),** with **c = 1, a = 20**, and **b = 0.5**, as shown below:

$$f(t) = \frac{1}{1 + 20e^{-0.5t}};\ t \geq 0$$



Answer the following questions:

1. Describe, in 2-3 sentences, the **notable differences and similarities** between the behaviors of a graph of *pure* exponential growth vs. that of *limited* logistic growth (shown above) as time increases.
2. Looking at the *limited* logistic growth graph above, **determine the approximate value of *f* (0). Show the calculation of the exact value of *f* (0),** and check the reasonableness of your answer with the value from the graph. What does the value of *f* (0) physically represent for a *limited* logistic growth model?
3. Looking at the *limited* logistic growth graph above, **as *t* increases, what value does *f* (*t*) approach?** What would we call the horizontal line drawn along this value in "math terms"? What does the line physically represent for a limited logistic growth model?
4. Let us consider a different logistic growth function: $g(t) = \frac{2}{1+10e^{-0.3t}};\ t \geq 0$ (**c = 2, a = 10, and b = 0.3**). *Without* graphing the function, **as t increases, what value does *g* (t) approach?** Justify your answer.

   Hint: Use your calculator to calculate values of g (t) for some *large* values of t. For example, calculate g (t) for t = 10, t = 20, t = 50, then t = 100, t = 150 and so on. **Try to notice a pattern in those values of g (t).** What value are they getting closer and closer to?

Part 2 – Application: the Spread of Influenza

In the second part of the activity, you will evaluate the effects of changing the values of ***a*** and ***b*** in a *limited* logistic growth model ***f*** **(*t*)** for an influenza outbreak, by plotting graphs for different scenarios (as outlined below). **To generate the plots, use the following free and easy to use online graphing tool:**

https://www.desmos.com/

Use this guidance in generating the plots and inserting them into your assignment document:

- Click **here** for a Desmos Tutorial for Beginners.
  Additional Information: There is a full collection of **interactive tours** and **video tutorials** available at **learn.desmos.com**

- Since $t \geq 0$ for logistic growth models, *ignore* the part of the plot for $t < 0$.

- **Here** is how to *type fractional expressions* in Desmos.

- To insert your plots into your document, see **this** tutorial. ***Alternatively***, simply **screen capture** your graph, crop it appropriately, and **paste** it into your document.

Model and Scenarios

Influenza (Google it) is a highly mutative virus that generally starts spreading in livestock populations. Eventually, a strain develops that spreads to humans among the farmers of chickens and pigs in third world countries that are not always able to maintain strict hygiene standards. New strains develop in the animal population fairly continuously, and the strongest of these varieties kill or greatly weaken their hosts. The World Health Organization (WHO) has a regular strategy to "trap" new strains mutating into new forms before cross-species variants develop. The Center for Disease Control (CDC) acts as coordinating agent in the US to gather information for WHO. They also are the principal developers of flu vaccines. Since the vast majority of people survive the flu, it isn't as critical to identify a new flu virus right away. In fact, a prospective new strain needs to first show an ability to spread rapidly before the government will spend a large sum of money to build a vaccine. For example, the deadliness of H5N1 (called avian flu) led CDC to set up a crash vaccine creation program for it in the early 2009. By 2010, it was incorporated into the mainstream fall season vaccine, along with two other strains that showed signs of severity in Asia in the summer and fall of 2008.

Interestingly, a flu virus generally takes longer to spread in the human population. For US population studies, where at least 250,000,000 people are vulnerable to a particular flu virus, as few as 300 people with flu-like symptoms may have their blood tested by the time a new virus is identified. So the value of the initial proportion of infected people = 300/250,000,000 = 0.0000012, a REALLY small decimal.

The primary effect of a vaccine is to reduce the susceptible population size. Using a limited logistic growth function as a model, the number of people in a given population that are ill with the flu at any given time can be determined. The limit of the number of people infected is the total size of the population. The *growth rate* of infection, however, can be limited by many factors. The effects of vaccination is one limiting factor, which causes a reduction in the vulnerable population, thus reducing the number of people infected. Vaccines also reduce the infectivity of a virus, so that its spread becomes even slower.

The following logistic growth model describes the percentage of a population, ***f*** **(*t*),** who have become ill with influenza *t* days after its initial outbreak.

$$\boldsymbol{f(t) = \frac{100}{1 + ae^{-bt}};\ t \geq 0}$$

Scenarios 1-3: changing the value of *b* in *f* (*t*)

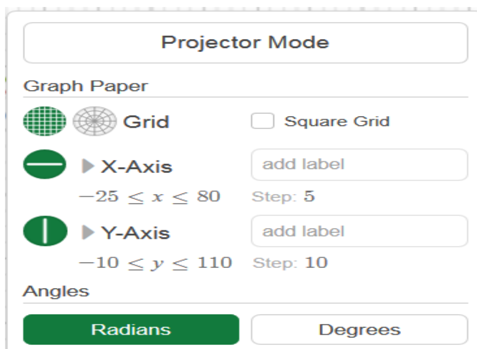
In **Desmos**, graph each of the following three scenarios on the ***same*** axes:

Scenario 1. In ***f*** **(*t*),** set ***a*** = 8 and ***b*** = 0.2. Enter ***f*** **(*t*)** for these settings into the line for **Graph 1.**

Scenario 2. In ***f*** **(*t*),** set ***b*** equal to ***twice*** the value in Scenario 1. Do not change ***a***. Enter ***f*** **(*t*)** for these settings into the line for **Graph 2.**

Scenario 3. In ***f*** **(*t*),** set ***b*** equal to ***half*** of the value in Scenario 1. Do not change ***a***. Enter ***f*** **(*t*)** for these settings into the line for **Graph 3.**

For the plot of the above graphs, use the following **axis settings** in Desmos. **Tutorial** on *how to adjust axis settings* in Desmos.



- **Insert this plot into your assignment document.** In your document, *clearly identify* the graphs.

Scenarios 4 and 5: changing the value of *a* in *f* (*t*)

In **Desmos,** REMOVE the equations in the lines for graphs **2** and **3** (scenarios 2 and 3), and *leave* the equation in the line for **graph 1** (scenario 1). Now, graph scenario 1 *and* the following two scenarios on the ***same*** axes:

Scenario 4. In ***f*** **(*t*),** set ***a*** equal to ***twice*** the value in Scenario 1. Do not change ***b***. Enter ***f*** **(*t*)** for these settings into the line for **Graph 4.**

Scenario 5. In ***f*** **(*t*),** set ***a*** equal to ***half*** of the value in Scenario 1. Do not change ***b***. Enter ***f*** **(*t*)** for these settings into the line for **Graph 5**.

For this plot, use the same **axis settings** as above.

- **Insert this plot into your assignment document.** In your document, clearly identify the graphs.

Answer the following questions:

5. On your 1st plot, compare the graphs of scenarios **1, 2, and 3**. *Briefly discuss, in 3-4 sentences, the differences or similarities in each of the following for the different values of **b**:*
   a. The percentage of the population infected on ***Day Zero*** (i.e., the value of f (0)). *Day Zero*, or $t = 0$, represents the beginning of the outbreak, at which time a certain percentage of the population is already infected.
   b. The time at which the limiting value of the function (i.e., the limit to the percentage of people infected) is approached.
   c. The time at which ***50%*** of the population became infected.
   d. The general shape of the curve.

6. On your 2nd plot, compare the graphs of scenarios **1, 4, and 5.** *Briefly discuss, in 3-4 sentences, the differences or similarities in each of the following for the different values of **a:***
   a. The percentage of the population infected on ***Day Zero*** (i.e., the value of f (0)).
   b. The time at which the limiting value of the function (i.e., the limit to the percentage of people infected) is approached.
   c. The time at which ***50%*** of the population became infected.
   d. The general shape of the curve.

7. In 3-4 sentences, summarize the physical meaning of the values of ***a*** and ***b*** within the logistic function for modeling the outbreak of influenza. In other words, what aspects of the trend of the outbreak are represented by the values of ***a*** and ***b***?

8. This question has two parts. Graphical solutions will <u>not</u> be accepted. To solve them, you need to recall *how to solve exponential equations*. See problem # 37 – 42 in Module 5 MML HW. $\boldsymbol{f(t) = \frac{100}{1+ae^{-.075t}};\ t \geq 0}$
   a) So, **c** = 100 and **b** = .075. Also, on day zero (t = 0), .04% of the population is already infected. Using this information, first find the value of **a**.
   b) Now determine **when** 50% of the population became infected.

9. Just like # 8, this question has two parts. Graphical solutions will <u>not</u> be accepted.

$$\boldsymbol{f(t) = \frac{100}{1+ae^{-bt}};\ t \geq 0}$$

   a) So, **c** = 100. Also, on day zero (t = 0), .02% of the population is already infected. Using this information, first find the value of **a**. (**Hint:** Notice, **b - value** is <u>not</u> given. A careful observation will show you that **b - value** is <u>not</u> needed to find the **a - value**)

b) Now calculate **what value of b** will result in 60% of the population being infected as of Day 300.