Math 109. Instructor: Chow
Homework #2. Due 3:00 pm on Monday, July 11, 2016.
REVISED

Problem 1: Prove, using logical argument from the definitions, that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

That is, prove that $x \in A \cup (B \cap C)$ if and only if $x \in (A \cup B) \cap (A \cup C)$.

HINT: You may use the fact that: 'P or (Q and R)' is logically equivalent to '(P or Q) and (P or R)'.

Problem 2:

(i) Prove: $\forall x \in \mathbb{Z}^+$, $\exists y \in \mathbb{Z}^+$, $y > 3x + 2$.

(ii) Prove: $\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$, $y > x^2 + 9$.

(iii) DIS Prove: $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $y < x^2$.

(iv) Disprove: $\forall n \in \mathbb{Z}^+$, $\exists m \in \mathbb{Z}^+$, $n \leq m^2 \leq n + 39$.

Problem 3:

(i) Prove: $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $2xy = x^3 + 2x^2$.

(ii) Disprove: $\exists x \in \mathbb{R}$, $\forall y \in \mathbb{R}$, $2xy > 1$.

(iii) Prove: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $\left(x^2 + 1\right) y = x^3 + 2x^2$.

(iv) Prove: $\forall x \in \mathbb{R} - \mathbb{Q}$, $\exists y \in \mathbb{R} - \mathbb{Q}$, $xy = 1$.

Problem 4: Let $n \in \mathbb{Z}$.

Let $P(n)$ be the statement: *There exists* $q \in \mathbb{Z}$ *such that* $n = 5q + 3$.

Let $Q(n)$ be the statement: *There exists* $p \in \mathbb{Z}$ *such that* $n^2 = 5p + 4$.

Prove that $P(n)$ implies $Q(n)$.

Hint: This is a direct argument, not a proof by induction.

Problem 5: Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. The mean value theorem says that there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$. Use the mean value theorem to prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \frac{1}{3}x^3 + 3x^2 + 10x$ is strictly increasing. (Recall that strictly increasing means that for any $a < b$, $f(a) < f(b)$.)

Problem 6: We say that $\lim_{x \to \infty} f(x) = \infty$ if for any $M \in \mathbb{R}$ there exists $N \in \mathbb{R}$ such that if $x \geq N$, then $f(x) \geq M$.

The intermediate value theorem says that if $f : [a, b] \to \mathbb{R}$ is a continuous function and if y is between $f(a)$ and $f(b)$, then there exists $x \in (a, b)$ such that $f(x) = y$.

(a) Define, analogously to the above, what it means for $\lim_{x\to-\infty} f(x) = -\infty$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. **Prove**: If $y \in \mathbb{R}$, then there exists $x \in \mathbb{R}$ such that $f(x) = y$.

Problem 7: Let $\mathbb{I}$ denote the irrational numbers. Define the function

$$f : \mathbb{I} \times \mathbb{I} \to \mathbb{R} \times \mathbb{R}$$

by

$$f(x, y) = \left(x + y, x^2 + y^2\right).$$

(i) Does there exist $(x, y) \in \mathbb{I} \times \mathbb{I}$ such that $f(x, y) \in \mathbb{I} \times \{1\}$?

(ii) Does there exist $(x, y) \in \mathbb{I} \times \mathbb{I}$ such that $f(x, y) = \mathbb{Q} \times \mathbb{I}$?

Problem 8: Given $n \in \mathbb{N}$, let $\mathbb{N}_n = \{1, 2, \ldots, n\} = \{a \in \mathbb{Z} \mid 1 \leq a \leq n\}$. Let X be a finite set. The number of elements in X, called the cardinality of X, is denoted by $|X|$. We have $|X| = n$ if and only if there exists a bijection $f : \mathbb{N}_n \to X$. Answer correctly the following (no need to prove anything).

(i) If $X \subseteq Y$, then how are $|X|$ and $|Y|$ related?

(ii) If $f : A \to B$ is an injection, then how are $|A|$ and $|B|$ related?

(iii) If $g : C \to D$ is a surjection, then how are $|C|$ and $|D|$ related?

(iv) If $h : E \to F$ is a bijection, then how are $|E|$ and $|F|$ related?

Problem 9: Do Problem 18 on p. 118.

Problem 10: Do Problem 20 on p. 118.

Remark: The original problems #11 and #12 on the inclusion-exclusion principle for 3 sets, have been moved to the 4th HW assignment.