Math 636 - Assignment 10 - Written Component
Due: Friday, July 22 at 4:00PM

1. Consider the function on $\mathbb{R}^2$ defined by

$$\left\langle \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\rangle = 2v_1w_1 - v_1w_2 - v_2w_1 + v_2w_2$$

   (a) Prove that this function is an inner product on $\mathbb{R}^2$.

   (b) Show that the standard basis vectors $\vec{e}_1, \vec{e}_2$ for $\mathbb{R}^2$ are not orthogonal under this inner product.

   (c) Find a basis for $\mathbb{R}^2$ that is orthogonal under this inner product.

2. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 4 & -2 \\ -2 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \right\}$ be a spanning set for a subspace $\mathbb{S}$ of $\mathbb{M}_{2\times 2}(\mathbb{R})$. Use the Gram-Schmidt procedure on $\mathcal{B}$ (in this order) to find an orthonormal basis for $\mathbb{S}$.

3. Let A and B be $n \times n$ matrices. If there exists an orthogonal matrix P such that $P^TAP = B$, then we say that A and B are **orthogonally similar**.

   (a) Prove that if A and B are orthogonally similar and are both invertible, then A^{-1} and B^{-1} are also orthogonally similar.

   (b) Prove that if C and D are orthogonally similar, then C^2 and D^2 are also orthogonally similar.

4. Let $\mathbb{V}$ be an n-dimensional real inner product space and let $\langle \vec{x}, \vec{y} \rangle$ and $[\vec{x}, \vec{y}]$ both be two different inner products on $\mathbb{V}$ Prove that there exists a linear mapping $L : \mathbb{V} \to \mathbb{V}$ such that

$$[L(\vec{x}), L(\vec{y})] = \langle \vec{x}, \vec{y} \rangle, \qquad \text{for all } \vec{x}, \vec{y} \in \mathbb{V}$$