Homework 2 Math 113 Summer 2016.

Due Thursday June 30th

Make sure to write your solutions to the following problems in complete English sentences. Solutions that are unreadable or incoherent will receive no credit. Provide complete justifications for all claims that you make. Problems will be of varying difficulty, and do not appear in any order of difficulty.

1. Let g be an element of order k in a group G.

   a) If $f\colon G \to H$ is a homomorphism, prove that the order of $f(g)$ divides k.

   b) If $f\colon G \to H$ is an isomorphism, prove that the order of $f(g)$ is equal to k.

2. Find all automorphisms of $\mathbb{Z}/4\mathbb{Z}$.

3. Does $\overline{k} \cdot x = x + k$ define an action of $\mathbb{Z}/n\mathbb{Z}$ on $\mathbb{R}$?

4. Let $U_3(\mathbb{Z}/2\mathbb{Z})$ be the group of matrices

$$U_3(\mathbb{Z}/2\mathbb{Z}) \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z}/2\mathbb{Z} \right\}$$

   with matrix multiplication as the binary operation.

   a) Show that $|U_3(\mathbb{Z}/2\mathbb{Z})| = 8$.

   b) Find an element R of order 4 and an element S of order 2 in $U_3(\mathbb{Z}/2\mathbb{Z})$, such that $SRS = R^{-1}$.

   c) Write the group D_8 as $\{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, where r is counterclockwise 90° rotation of the plane and s is a reflection. Consider the map

$$f : D_8 \to U_3(\mathbb{Z}/2\mathbb{Z})$$

   which sends $s^i r^j$ to $S^i R^j$. This defines a homomorphism (you do not need to prove this). Prove that f is actually an isomorphism.

5. (Important, and useful for the following exercises!) Prove that "a homomorphism is determined by what it does to the generators", in the following sense. Suppose that a group G is generated by some subset B. Suppose f_1 and f_2 are two homomorphisms to some other group H such that $f_1(g) = f_2(g)$ for all g in B. Prove that $f_1 = f_2$. [CAUTION: it does not follow, as is the case in linear algebra, that we can define a homomorphism simply by specifying where to send the generators; one has to be careful about possible relations that the generators may satisfy]

6. Prove that there are no nontrivial homomorphisms[1] from D_{10} to $\mathbb{Z}/5\mathbb{Z}$. [Hint: use problem 1]

[1]The **trivial homomorphism** from G to H is the map $f(g) = e_H$ for all $g \in G$. A homomorphism is **nontrivial** if it is not this one.

7. In the dihedral group D_{12} (symmetries of a regulator hexagon centered at the origin with two of its vertices on the x-axis) , describe the subgroup H consisting of transformations which fix the line L given by $y = \sqrt{3}x$ (meaning they leave L unchanged). Find the *right*-coset of this subgroup which takes the x-axis to L. In other words, find an element $g \in D_{12}$ such that the elements of the *right*-coset Hg are all those symmetries which take the x-axis to L.

8. Using Lagrange's theorem, determine all pairs m, n of positive integers for which there exists a nontrivial homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$.

9. Find all possible actions on the group $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/3\mathbb{Z}$.

10. Finish the "converse" part of Lemma 6.1.2 in the notes, namely: let α be a homomorphism $G \to \mathrm{Perm}(S)$; use α to define an action of G on S.