Matroids

A matroid is an ordered pair (S, I), in which S is a finite and nonempty set, while I is a nonempty collection of subsets of S, called the *independent* subsets of S, so that the following two properties hold:

1: if A belongs to I and B $\subseteq$A, then B belongs to I (hereditary property);

2: if A and B belong to I and |A| > |B|, then there is an x in A \ B such that B $\cup$ {x} belongs to I (exchange property).

This is a good example of a matroid, which is called a *graphic matroid*. Fix an arbitrary undirected graph G = (V,E). Define I as the set of all acyclic subsets of E, so, the set of all subsets A of E such that the graph (V,A) is acyclic. (E, I) is a matroid.

An independent set A of a matroid is *maximal* if it's not properly contained within any other independent set. In the case of a graphic matroid, the maximal independent sets correspond to the spanning forests of the graph G. (If G is connected, then the maximal independent sets correspond to the spanning trees of G.) It is not difficult to prove that every spanning forest of a given undirected graph has the same number of edges. (More precisely, if a graph G = (V,E) has k connected components, then every spanning forest of G has |V| - k edges.) This fact can be seen as a special case of the following result concerning arbitrary matroids, which is an easy consequence of the exchange property: All maximal independent subsets of a given matroid have the same cardinality.

The Matroid Greedy Algorithm

A *weighted* matroid is a matroid (S, I) where each element of S has an associated weight. A fundamental matroid optimization problem is the following: Given a weighted matroid, determine a minimum-weight maximal independent set. (The weight of an independent set is dened as the sum of the weights of the elements of the set.)

In the special case of a graphic matroid, the problem of determining a minimum-weight maximal independent set corresponds to the minimum spanning forest problem. Recall that Kruskal's algorithm can be used to solve the minimum spanning forest problem. Can Kruskal's algorithm be generalized to solve the minimum-weight maximal independent set problem for arbitrary matroids? Yes, Kruskal's algorithm is a special case of the *matroid greedy algorithm*, which does just that. The matroid greedy algorithm works as follows.

First, we sort the elements of S in nondecreasing order of weight, breaking ties arbitrarily, and we initialize A to the empty set, which is guaranteed to be independent. (That the empty set is independent follows from the hereditary property, and the requirement that there is at least one independent set.) Then, we consider each x in S in turn, in the order established by the sort. When we consider x, we add it to A if and only if A$\cup$ {x} is independent. After considering all of the elements of S, we claim that the set A is a minimum-weight maximal independent set. The proof of this claim is similar to the proof of correctness of Kruskal's algorithm.

The same framework can be used to determine a maximum-weight maximal independent set. The only difference is that we sort the elements of S in nonincreasing order, rather than in nondecreasing order.

Recall that one of the consequences of our analysis of Kruskal's algorithm was that every MST has the same distribution of edge weights. This result generalizes to arbitrary weighted matroids: Given a weighted matroid M, if A and B are two maximum-weight (resp., minimum-weight) maximal independent sets of M, then for any weight z, the number of elements in A with weight z is equal to the number of elements in B with weight z.

A Matroid Based on an Edge-Weighted Bipartite Graph

Let G = (U, V,E) be an undirected bipartite graph, where U denotes the set of “left" vertices, V denotes the set of “right" vertices, and each edge e in E has one endpoint in U and one endpoint in V . Further assume that each edge e in E has an associated nonnegative weight weight(e). A matching of G is a subset M of E such that each vertex in $U \cup V$ is incident on at most one edge of M. Given a matching M of G, each vertex of G that is incident on some edge of M is said to be matched by M; the remaining vertices are unmatched. The weight of a matching M of G, denoted weight(M), is defined as $\sum_{e \in M}$ weight(e). A maximum-weight matching (MWM) of G is a matching M of G such that weight(M) ≥ weight(M’) for all matchings M’ of G.

Problem 1:
Let G = (U, V,E) be a bipartite graph with nonnegative edge weights. Let us say that a subset A of U is independent if some MWM of G matches every vertex in A. Let $\mathcal{A}$ denote the set of all independent subsets of U. **Prove that (U,$\mathcal{A}$) is a matroid.**

While the definition of the matroid (U,$\mathcal{A}$) of the above problem is based on a bipartite graph where the edges have weights, the elements of U do not have weights, so this is not a weighted matroid. Now assume that each vertex u in U has an associated real priority, denoted priority(u). Furthermore, for any subset U’ of U, let us dene the priority of U’, denoted priority(U’), as $\sum_{u \in U'}$priority(u). The matroid greedy algorithm (with the priorities playing the role of the weights) can be used to determine a maximum-priority maximal independent set of matroid (U,$\mathcal{A}$). In fact, just as Kruskal's algorithm can generate (by appropriate tie-breaking) any MST of a given connected, edge- weighted graph, the matroid greedy algorithm can generate (by appropriate tie-breaking with respect to the priorities of the left vertices) any maximum-priority maximal independent set of matroid (U,$\mathcal{A}$). Given a bipartite graph G = (U, V, E) with nonnegative edge weights, and where each vertex u in U has an associated priority, we say that an MWM M of G is greedy if the set of left vertices matched by M is a maximum-priority maximal independent set of matroid (U,$\mathcal{A}$).
We remark that it is easy to give an equivalent characterization of the set of greedy MWMs that does not rely on matroid terminology. First, we can argue that the set of left vertices matched by an MWM M of G is a maximal independent set of the matroid (U;A) if and only if M is a

maximum-cardinality MWM (MCMWM) of G. Next, for any MWM M of G, dene priority(M) as the sum of the priorities of the left vertices matched by M. Using this definition, we find that an MWM of G is greedy if and only if it is a maximum-priority MCMWM of G.

Problem 2:
Let G = (U, V, E) be a bipartite graph with nonnegative edge weights and where each left vertex has a real priority. Let U_0 and U_1 be subsets of U such that $U_0 \subseteq U_1$, let u be a vertex in U_0, and for i in {0, 1}, let G_i denote the subgraph of G induced by $U_i \cup V$. **Prove that if u is not matched in any greedy MWM of G_0, then u is not matched in any greedy MWM of G_1.**