Question 2 (40 points)

a) (10 points)

Solve the recurrence

$$t(n) = t\left(\frac{n}{2}\right) + log_2 n\ , \quad \text{such that } t(1) = 0.$$

Assume in your solution that n is a power of 2, that is, $n = 2^m$ so $m = log_2 n$.

b) (10 points)

Verify that your answer in (a) is correct using a proof by mathematical induction, namely perform induction on the variable m.

c) (10 points)

If two positive integers n_1 and n_2 have N_1 and N_2 decimal digits, respectively, then computing their product $n_1 * n_2$ using the grade school multiplication algorithm has time complexity $O(N_1\, N_2)$.

Question: For a given positive integer n, what is the $O(\ )$ time complexity of computing n^n, that is, n to the power n ?

Hints:

- The answer is not $O(N^n)$, i.e. $O((log_{10} n)^{\boldsymbol{n}})$. Such an answer would ignore the fact(s) that you need multiply n repeatedly by itself, and that the result has an increasing number of digits.
- Answer this question using arguments and concepts that were covered prior to lecture 14. i.e. Do not use the lecture 14 formal definition of big O here.
- In answering the question, do *not* attempt to use any tricks such as computing n^8 by first computing n^4 and then squaring the result. In fact, such tricks do not speed up the computation of n^n in a big O sense. I will discuss this later in the course, time permitting.
- The number of digits N to represent a positive integer n in base 10 is floor$(log_{10} n) + 1$, but to answer the question, you should approximate this formula as $log_{10}\, n$. See lecture notes 2 pages 5,6 where a similar formula is derived for base 2.

d) (10 points)

Let $t(n) = \sqrt{n^2 + 100\,n} \; - \; n.$

Use the formal definition of $O(\,)$ from lecture 14 to show that $t(n)\; is\; O(1)$.

What to submit ?

For Question 2(a-d), submit one PDF file **FirstnameLastname.pdf** to the A2Q2 mycourses assignment folder. It will be graded separately from Question 1.

We do *not* require that you typeset your solution. If you do typeset it for fun, that would be fine, and we do encourage you to learn to use typesetting software such as latex. However, learning how to use this software may not the best way for you to spend your time these days.

We do insist that you submit a PDF, though. For example you could do the following:

- write out the solutions neatly by hand and scan them into a single PDF using McGill uPrint
- take a cell phone photo of each page, and insert the images into an MS Word or OpenOffice doc, and save the doc as a PDF.

All symbols and numbers in your document must be clearly legible. Moreover, the logical reasoning of your proofs must be correct and clearly stated. The point of these questions for you to demonstrate that you understand and can express the steps involved. The level of explanations in your solutions should be comparable to the examples in the lecture notes and the exercises.