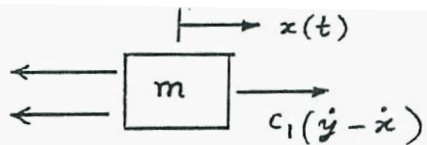**Homework 2**

Textbook Problems:

(1) Work through the 1st problem for Section **3.6** (**Figure 3.14**)
(2) Work through Example **3.4**
(3) Work through the 1st problem for Section **3.7** (**Figure 3.19**)
(4) Work through Example **3.6**
(5) Problem **3.35** (Calculate only the steady state (equilibrium state) response)
(6) Problem **3.45** (Steady state response)
(7) Problem **3.71** (**3.63** in the 5th Edition textbook; calculate steady state response)
(8) Problem **4.8** (Steady state response)
(9) Problem **4.12** (Steady state response)

*Notes:
(1) Lecture note 3&4 tells us how a standard mass-spring-damper system responds to a force. In this HW, you will start with mechanical systems that look somehow different from a standard mass-spring-damper system, but can be described by dynamics equations that are equivalent to that of a standard mass-spring-damper system. Work on (1) to (4) to learn more.

(2) Under a harmonic force the particular solution x_p is usually the steady state (equilibrium) motion that does not vanish. In contrast, the homogeneous solution x_h is usually the transient state solution that vanishes when a friction exists (zeta > 0).

3.35

(a) Equation of motion of mass:

$$m\ddot{x} = c_1(\dot{y} - \dot{x}) - c_2\dot{x} - k_2 x$$

i.e.,

$$m\ddot{x} + (c_1 + c_2)\dot{x} + k_2 x = c_1\dot{y} = -c_1 \omega Y \sin \omega t$$

(b)
$$x_p(t) = \frac{-(c_1 \omega Y / k_2)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi)$$

where $r = \omega/\omega_n$, $\zeta = (c_1 + c_2)\omega/(2 r k)$ and $\phi = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$.

(c) steady-state force transmitted to point P:

$$= k_2 x_p + c_2 \dot{x}_p$$

$$= \frac{-(c_1 \omega Y)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}\left\{\sin(\omega t - \phi) + \frac{c_2 \omega}{k_2}\cos(\omega t - \phi)\right\}$$

3.45

Start this problem by the same procedure as for the 1ˢᵗ problem of section 3.6, "Response of a Damped System Under the Harmonic Motion of the Base". From there you receive the ratio X/Y, which helps solve the problem as follows:

$$\frac{X}{Y} = \left[\frac{k^2 + c^2\,\omega^2}{\left(k - m\,\omega^2\right)^2 + c^2\,\omega^2} \right]^{\frac{1}{2}} \quad .sp$$

or

$$\frac{10^{-6}}{Y} = \left[\frac{\left(10^6\right) + \left(10^3\,(200\,\pi)\right)^2}{\left\{ \left(10^6 - \left(\frac{5000}{9.81}\right)(200\,\pi)^2\right)^2 + \left\{\left(10^3\right)(200\,\pi)\right\}^2\right\}} \right]^{\frac{1}{2}}$$

or $Y = 169.5294\,(10^{-6})\,m$

3.71

Start this problem by the same procedure as for the 1st problem of section 3.7,"Response of a Damped System Under Rotating Unbalance". From there you receive force term.

Equation of motion: $M\ddot{x} + c\dot{x} + kx = m\,e\,\omega^2 \sin \omega t$

where $\omega = \dfrac{3000\,(2\pi)}{60} = 314.16$ rad/sec, $M = 100$ kg, $c = 2000$ N-s/m, $k = 10^6$ N/m, $m = 0.1$ kg and $e = r = 0.1$ m. Steady state response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

$$\text{where} \quad X = \frac{m\,e\,\omega^2}{\left[(k - M\omega^2)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}}$$

$$= \frac{0.1\,(0.1)\,(314.16^2)}{\left[\left\{10^6 - 100\,(314.16^2)\right\}^2 + (2000\,(314.16))^2\right]^{\frac{1}{2}}} = 110.9960\,(10^{-6})\,\text{m}$$

$$\text{and} \quad \phi = \tan^{-1}\left(\frac{c\,\omega}{k - M\omega^2}\right) = \tan^{-1}\left(\frac{2000\,(314.16)}{10^6 - 100\,(314.16^2)}\right)$$

$$= -0.07072 \text{ rad} = -4.0520°$$

4.8

Base motion can be represented by Fourier series as (from Example 1.19):

$$y(t) = \frac{Y}{\pi} \left[\frac{\pi}{2} - \left\{ \sin \omega t + \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t + \cdots \right\} \right] \qquad (1)$$

Equation of motion of mass:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \qquad (2)$$

Since y(t) is composed of several terms, the solution of Eq. (2) can be found by superposingg the solutions corresponding to each of the terms appearing in Eq. (1). When y(t) = Y/2, constant, equation of motion becomes:

$$m\ddot{x} + c\dot{x} + kx = \frac{kY}{2} = \text{constant} \qquad (3)$$

The steady state solution of Eq. (3) is given by

$$x(t) = \frac{Y}{2} \qquad \leftarrow \text{Refer to "Case 2" in the lecture note} \qquad (4)$$

$$(5)$$

When y(t) = A sin Ω t, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin(\Omega t - \phi) \qquad (6)$$

$$\text{where } A = \left[\frac{1 + (2\varsigma r)^2}{(1 - r^2)^2 + (2\varsigma r)^2} \right]^{\frac{1}{2}} \qquad (7)$$

$$\phi = \tan^{-1}\left(\frac{2\varsigma r^3}{1 + r^2(4\varsigma^2 - 1)} \right) \qquad (8)$$

$$\text{and } r = \frac{\Omega}{\omega_n}$$

4.12

$k = 5 \times 10^6$ N/m , $m = 750$ kg , $c = 0$

Equation of motion of a mass subjected to base excitation:

$$m\ddot{x} + kx = k\, y(t)$$

$$= \frac{k}{\pi} + \frac{k}{2}\sin 2\pi t$$

$$- \frac{2k}{\pi}\left\{ \frac{\cos 4\pi t}{1(3)} + \frac{\cos 8\pi t}{3(5)} + \frac{\cos 12\pi t}{5(7)} + \cdots \right\}$$

$$(1)$$

Response under different components of forcing function:

(i) When

$$m\ddot{x} + kx = \frac{k}{\pi} \qquad (2)$$

Let $\quad x_p(t) = A_0 = $ constant $\qquad (3)$

Substitute Eq. (3) into Eq. (2):

$$k A_0 = \frac{k}{\pi} \qquad \text{or} \qquad A_0 = \frac{1}{\pi}$$

$$\therefore \; x_p(t) = \frac{1}{\pi} \qquad (4)$$

(ii) When

$$m\ddot{x} + kx = \frac{k}{2}\sin 2\pi t = A_1 \sin \omega t \qquad (5)$$

with $\quad A_1 = \frac{k}{2}$ and $\omega = 2\pi$

Let $\quad x_p(t) = X \sin \omega t \qquad (6)$

Substitute Eq. (6) into Eq. (5);

$$-m X \omega^2 \sin \omega t + kX \sin \omega t = A_1 \sin \omega t$$

or

$$X = \frac{A_1}{-m\omega^2 + k} = \frac{k}{2(k - m\omega^2)} = \frac{1}{2\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

where $\quad \omega_n = \sqrt{\frac{k}{m}} = \left(\frac{5 \times 10^6}{750}\right)^{\frac{1}{2}} = 81.6496$ rad/s

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{6.2832}{81.6496}\right)^2 = (0.07695)^2 = 0.005922$$

$$X = \frac{1}{2(1-0.005922)} = 0.5030 \text{ m}$$

$$\therefore x_p(t) = 0.5030 \sin 2\pi t \quad \text{m} \tag{7}$$

(iii) When

$$m\ddot{x} + kx = -\frac{2k}{\pi} \frac{\cos \omega_2 t}{3} = A_2 \cos \omega_2 t \tag{8}$$

with $A_2 = -\frac{2k}{3\pi}$ and $\omega_2 = 4\pi$

Let

$$x_p(t) = X \cos \omega_2 t \tag{9}$$

Substitute Eq. (9) in Eq. (8):

$$-m\omega_2^2 X + kX = A_2$$

or

$$X = \frac{A_2}{k - m\omega_2^2} = -\frac{2k}{3\pi(k - m\omega_2^2)}$$

$$= -\frac{2}{3\pi\left(1 - \frac{\omega_2^2}{\omega_n^2}\right)} = -\frac{2}{3\pi\left(1 - \left(\frac{12.5664}{81.6496}\right)^2\right)}$$

$$= -\frac{2}{3\pi}\left(\frac{1}{1 - 0.02369}\right) = -0.2173 \text{ m}$$

$$\therefore x_p(t) = -0.2173 \cos 4\pi t \tag{10}$$

(iv) When

$$m\ddot{x} + kx = -\frac{2k}{\pi} \frac{\cos i\pi t}{\left(\frac{i}{2} - 1\right)\left(\frac{i}{2} + 1\right)} \quad ; \quad i = 4, 8, 12, \cdots$$

← Note that here 'i' is a variable, not the imaginary number i

$$= -\frac{8k}{\pi} \frac{\cos i\pi t}{(i - 2)(i + 2)} \quad ; i = 4, 8, 12, \cdots \tag{11}$$

Substitute Eq. (12) in Eq. (11):

$$- \omega_i^2 \, m \, X + k \, X = A_i$$

or

$$X = \frac{A_i}{k - m \, \omega_i^2} = \frac{(A_i/k)}{\left\{ 1 - \left(\frac{\omega_i}{\omega_n} \right)^2 \right\}}$$

$$= - \frac{\left\{ \dfrac{8k}{k \, \pi \, (i-2)(i+2)} \right\}}{\left\{ 1 - \dfrac{i^2 \, \pi^2}{(81.6496)^2} \right\}}$$

$$= - \frac{8}{\pi \left\{ 1 - \dfrac{i^2 \, (9.8696)}{(81.6496)^2} \right\} (i-2)(i+2)}$$

$$= - \frac{2.54647}{\left(1 - 0.0014804 \, i^2 \right) (i-2)(i+2)} \qquad (12)$$

For $i = 4$:

$$X = - \frac{2.54647}{(1 - 0.02364864)(12)} = - 0.2173 \ m$$

For $i = 8$:

$$X = - \frac{2.54647}{(1 - 0.09474)(6)(10)} = - 0.04688 \ m$$

For $i = 12$:

$$X = - \frac{2.54647}{(1 - 0.21318)(10)(14)} = - 0.02312 \ m$$

etc. Hence steady state displacement of the car is:

$$x_p(t) = \frac{1}{\pi} + 0.5030 \, \sin 2\pi t - 0.2173 \, \cos 4\pi t$$

$$- 0.04688 \, \cos 8\pi t - 0.02312 \, \cos 12\pi t - \cdots \quad m \quad (13)$$