Homework #1

Textbook Ch 1: Work on Problems 109, 110, and 111. Try on problem 117.

You do not need to complete problem 117 (no exam will be given like that), but I put it here for your reference about how to do Fourier analysis numerically

Solution 1.109

1.109

$$x(t) = \begin{cases} \dfrac{2At}{\tau}, & 0 \le t \le \dfrac{\tau}{2} \\ -\dfrac{2At}{\tau} + 2A, & \dfrac{\tau}{2} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2} \frac{2At}{\tau}\cdot dt + \int_{\tau/2}^{\tau}\left(-\frac{2At}{\tau} + 2A\right)\cdot dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\cdot\frac{t^2}{2}\bigg|_0^{\tau/2} - \frac{2A}{\tau}\cdot\frac{t^2}{2}\bigg|_{\tau/2}^{\tau} + 2A\cdot t\bigg|_{\tau/2}^{\tau}\right]$$

$$= \frac{2}{\tau}\left[\frac{A\tau}{4} - \frac{3A\tau}{4} + A\tau\right] = A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt$$

$$= \frac{2}{\tau}\left[\int_0^{\frac{\tau}{2}} \frac{2A}{\tau}t\cos n\omega t\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}t + 2A\right)\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\sin n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

As $\tau = \dfrac{2\pi}{\omega}$,

$$a_n = \frac{\omega}{\pi}\left[\frac{A\omega}{\pi n^2\omega^2}\cos n\pi - \frac{A\omega}{\pi n^2\omega^2} - \frac{A\omega}{\pi n^2\omega^2}\cos 2\pi n + \frac{A\omega}{\pi n^2\omega^2}\cos n\pi\right]$$

$$= \frac{2A}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} -\dfrac{4A}{n^2\pi^2}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2}\frac{2A}{\tau}t\sin n\omega t\,dt\right.$$

$$\left. + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}t + 2A\right)\sin n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[-\frac{A}{n\omega}\cos n\pi + \frac{2A}{n\omega}\cos 2n\pi - \frac{A}{n\omega}\cos n\pi - \frac{2A}{n\omega}\cos 2n\pi + \frac{2A}{n\omega}\cos n\pi\right] = 0$$

$$\therefore\ x(t) = \frac{A}{2} - \frac{4A}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos n\omega t$$

Solution 1.110

1.110

$$x(t) = \begin{cases} \dfrac{4At}{\tau} & , \quad 0 \le t \le \dfrac{\tau}{4} \\ -\dfrac{4At}{\tau} + 2A & , \quad \dfrac{\tau}{4} \le t \le \dfrac{3\tau}{4} \\ \dfrac{4At}{\tau} - 4A & , \quad \dfrac{3\tau}{4} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\,\frac{t^2}{2}\Big|_0^{\tau/4} + \left(-\frac{4A}{\tau}\,\frac{t^2}{2} + 2At\right)_{\tau/4}^{3\tau/4} + \left(\frac{4A}{\tau}\,\frac{t^2}{2} - 4At\right)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t\cos n\omega t\cdot dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t\cos n\omega t\,dt + 2A\int_{\tau/4}^{3\tau/4}\cos n\omega t\,dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t\cos n\omega t\,dt - 4A\int_{3\tau/4}^{\tau}\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/4} - \frac{4A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/4}^{3\tau/4} + 2A\left(\frac{\sin n\omega t}{n\omega}\right)_{\tau/4}^{3\tau/4} + \frac{4A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{3\tau/4}^{\tau} - 4A\left(\frac{\sin n\omega t}{n\omega}\right)_{3\tau/4}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[\sin\frac{n\pi}{2}\left(\frac{A}{n\omega} + \frac{A}{n\omega} - \frac{2A}{n\omega}\right) + \cos\frac{n\pi}{2}\left(\frac{2A}{\pi n^2\omega} + \frac{2A}{\pi n^2\omega}\right) + \sin\frac{3n\pi}{2}\left(-\frac{3A}{n\omega} + \frac{2A}{n\omega} - \frac{3A}{n\omega} + \frac{4A}{n\omega}\right) + \cos\frac{3n\pi}{2}\left(-\frac{2A}{\pi n^2\omega} - \frac{2A}{\pi n^2\omega}\right) + \cos 2\pi n\left(\frac{2A}{\pi n^2\omega}\right) - \cos 0\left(\frac{2A}{\pi n^2\omega}\right)\right] = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t\sin n\omega t\,.dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t\sin n\omega t\,.dt + 2A\int_{\tau/4}^{3\tau/4}\sin n\omega t\,.dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t\sin n\omega t\cdot dt - 4A\int_{3\tau/4}^{\tau}\sin n\omega t\,.dt\right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_0^{\tau/4 = \frac{\pi}{2\omega}} - \frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_{\frac{\tau}{4} = \frac{\pi}{2\omega}}^{\frac{3\tau}{4} = \frac{3\pi}{2\omega}} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\frac{\tau}{4} = \frac{\pi}{2\omega}}^{\frac{3\tau}{4} = \frac{3\pi}{2\omega}} + \frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t\right.$$

$$- \frac{t}{n\omega} \cos n\omega t \Big|_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} - 4A \left(- \frac{\cos n\omega t}{n\omega}\right)\Big|_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} \Bigg]$$

$$= \frac{4A}{\pi^2 n^2}\left(\sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8A}{\pi^2 n^2}(-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \ x(t) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\sin n\omega t}{n^2}$$

Solution 1.111

1.111 $x(t) = A\left(1 - \frac{t}{\tau}\right) \quad , \quad 0 \le t \le \tau$

$$a_0 = \frac{2}{\tau}\int_0^\tau x(t)\,dt = \frac{2A}{\tau}\int_0^\tau \left(1 - \frac{t}{\tau}\right) dt = \frac{2A}{\tau}\left(t - \frac{t^2}{2\tau}\right)_0^\tau = A$$

$$a_n = \frac{2}{\tau}\int_0^\tau x(t)\cos n\omega t\,dt = \frac{2A}{\tau}\left(\frac{\sin n\omega t}{n\omega} - \frac{t}{\tau}\frac{\sin n\omega t}{n\omega} - \frac{\cos n\omega t}{\tau n^2\omega^2}\right)_0^{2\pi/\omega}$$

$$= 0$$

$$b_n = \frac{2}{\tau}\int_0^\tau x(t)\sin n\omega t\,dt = \frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega} + \frac{t}{\tau}\frac{\cos n\omega t}{n\omega} - \frac{\sin n\omega t}{\tau n^2\omega^2}\right)_0^{2\pi/\omega}$$

$$= \frac{A}{\pi n}$$

$$\therefore \; x(t) = \frac{A}{2} + \frac{A}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\omega t}{n}$$

Solution 1.117

1.117

i	t_i	x_i	$n=1$: $x_i \cos \frac{2\pi t_i}{0.6}$	$n=1$: $x_i \sin \frac{2\pi t_i}{0.6}$	$n=2$: $x_i \cos \frac{4\pi t_i}{0.6}$	$n=2$: $x_i \sin \frac{4\pi t_i}{0.6}$	$n=3$: $x_i \cos \frac{6\pi t_i}{0.6}$	$n=3$: $x_i \sin \frac{6\pi t_i}{0.6}$
1	0.025	9.00	8.69	2.33	7.79	4.50	6.36	6.36
2	0.050	17.00	14.72	8.50	8.50	14.72	0.00	17.00
3	0.075	23.00	16.26	16.26	0.00	23.00	−16.26	16.26
4	0.100	25.00	12.50	21.65	−12.50	21.65	−25.00	0.00
5	0.125	26.00	6.73	25.11	−22.52	13.00	−18.38	−18.38
6	0.150	28.00	0.00	28.00	−28.00	0.00	0.00	−28.00
7	0.175	33.00	−8.54	31.88	−28.58	−16.50	23.33	−23.33
8	0.200	35.00	−17.50	30.31	−17.50	−30.31	35.00	0.00
9	0.225	34.00	−24.04	24.04	0.00	−34.00	24.04	24.04
10	0.250	29.00	−25.11	14.50	14.50	−25.11	0.00	29.00
11	0. 75	24.00	−23.18	6.21	20.78	−12.00	−16.97	16.97
12	0.300	26.00	−26.00	0.00	26.00	0.00	−26.00	0.00
13	0.325	32.00	−30.91	−8.28	27.71	16.00	−22.63	−22.63
14	0.350	40.00	−34.64	−20.00	20.00	34.64	0.00	−40.00
15	0.375	18.00	−12.73	−12.73	0.00	18.00	12.73	−12.73
16	0.400	8.00	−4.00	−6.93	−4.00	6.93	8.00	0.00
17	0.425	−5.00	1.29	4.83	4.33	−2.50	−3.54	−3.54
18	0.450	−14.00	0.00	14.00	14.00	0.00	0.00	−14.00
19	0.475	−28.00	−7.25	27.05	24.25	14.00	19.80	−19.80
20	0.500	−37.00	−18.50	32.04	18.50	32.04	37.00	0.00
21	0.525	−33.00	−23.33	23.33	0.00	33.00	23.33	23.34
22	0.550	−29.00	−25.11	14.50	−14.50	25.11	0.00	29.00
23	0.575	−22.00	−21.25	5.69	−19.05	11.00	−15.56	15.56
24	0.600	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{i=1}^{24}(\ )$		239.00	−241.90	282.30	39.72	147.18	45.26	−4.88
$\frac{1}{12}\sum_{i=1}^{24}(\ )$		19.92	−20.16	23.53	3.31	12.26	3.77	−0.41