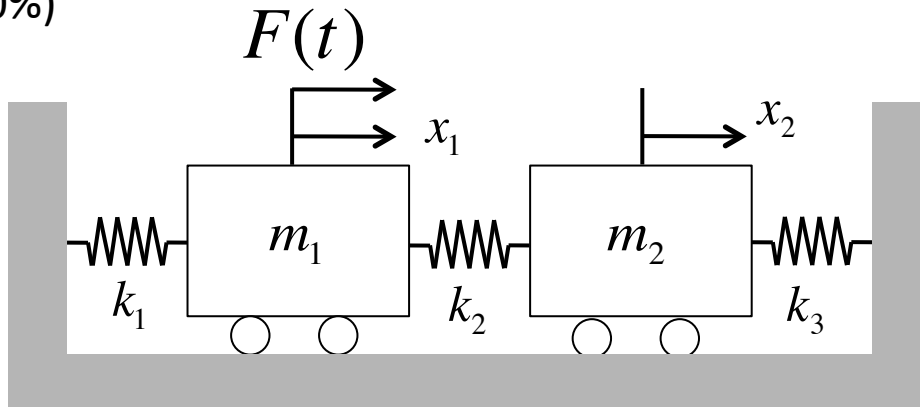


Midterm II & Solutions

(1) (60%)



Given the above mass-and-spring system. $F(t) = \sin(3t)$ is applied to mass m_1 . The system has zero initial conditions, $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

(a) Derive the dynamics equations for displacements x_1 and x_2 , using the following numbers:

$m_1 = 1$, $m_2 = 1$, $k_1 = 2 - \sqrt{2}$, $k_2 = \sqrt{2}$, $k_3 = 3 - \sqrt{2}$, $F(t) = \sin(3t)$. Identify matrices $[m]$ and $[k]$

(b) Calculate the natural frequencies ω_1 and ω_2 .

(c) Calculate the *eigenvectors* $X^{(1)}$ and $X^{(2)}$ of the system. Make sure to normalize the eigenvectors with respect to $[m]$.

(d) Convert the dynamics equations into equations of generalized displacement q_1 and q_2 , where $[x] = [X][q]$.

(e) Solve q_1 and q_2

(f) Solve x_1 and x_2

Hint: You need this equation: $\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

(a)

$$\begin{aligned} \ddot{x}_1 + 2x_1 - \sqrt{2}x_2 &= \sin(3t) \\ \ddot{x}_2 - \sqrt{2}x_1 + 3x_2 &= 0 \end{aligned} \Rightarrow [m] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, [k] = \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix}$$

(b)

$$-\omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X + \begin{bmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix} X = 0 \Rightarrow \begin{bmatrix} 2 - \omega^2 & -\sqrt{2} \\ -\sqrt{2} & 3 - \omega^2 \end{bmatrix} X = 0$$

$$\omega^4 - 5\omega^2 + 4 = 0 \Rightarrow \omega^2 = 1 \text{ or } 4 \Rightarrow \omega_1 = 1, \omega_2 = 2$$

(c)

$$\omega_1 = 1 \Rightarrow \begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix} = 0; \text{ let } X_1^{(1)} = a \Rightarrow X^{(1)} = \begin{bmatrix} a \\ a/\sqrt{2} \end{bmatrix}$$

$$\text{Let } X^{(1)T} [m] X^{(1)} = 1 \Rightarrow a^2 \begin{bmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1/\sqrt{2} \end{bmatrix} = 1 \Rightarrow a^2 \cdot \frac{3}{2} = 1 \Rightarrow a = \sqrt{\frac{2}{3}} \Rightarrow X^{(1)} = \begin{bmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{bmatrix}$$

$$\omega_2 = 2 \Rightarrow \begin{bmatrix} -2 & -\sqrt{2} \\ -\sqrt{2} & -1 \end{bmatrix} \begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix} = 0; \text{ let } X_1^{(2)} = b \Rightarrow X^{(2)} = \begin{bmatrix} b \\ -\sqrt{2}b \end{bmatrix}$$

$$\text{Let } X^{(2)T} [m] X^{(2)} = 1 \Rightarrow b^2 \begin{bmatrix} 1 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\sqrt{2} \end{bmatrix} = 1 \Rightarrow b^2 \cdot 3 = 1 \Rightarrow b = \sqrt{\frac{1}{3}} \Rightarrow X^{(2)} = \begin{bmatrix} \sqrt{1/3} \\ -\sqrt{2/3} \end{bmatrix}$$

(d)

$$[X] = \begin{bmatrix} X^{(1)} & X^{(2)} \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} & \sqrt{1/3} \\ \sqrt{1/3} & -\sqrt{2/3} \end{bmatrix},$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = [X]^T [F] = \begin{bmatrix} \sqrt{2/3} & \sqrt{1/3} \\ \sqrt{1/3} & -\sqrt{2/3} \end{bmatrix} \begin{bmatrix} \sin(3t) \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} \sin(3t) \\ \sqrt{1/3} \sin(3t) \end{bmatrix}$$

$$\Rightarrow \text{General displacement equation: } \ddot{q}_j + \omega_j^2 q_j = Q_j, j = 1, 2$$

$$\Rightarrow \ddot{q}_1 + q_1 = \sqrt{2/3} \sin(3t), \ddot{q}_2 + 4q_2 = \sqrt{1/3} \sin(3t)$$

(e)

$$\begin{aligned} \frac{1}{\omega_j} \int_0^t \sin(\omega \tau) \sin(\omega_j(t - \tau)) d\tau &= \frac{1}{2\omega_j} \int_0^t [\cos(\omega \tau - \omega_j t + \omega_j \tau) - \cos(\omega \tau + \omega_j t - \omega_j \tau)] d\tau \\ &= \frac{1}{2\omega_j} \left[\frac{\sin((\omega + \omega_j)\tau - \omega_j t)}{\omega + \omega_j} - \frac{\sin((\omega - \omega_j)\tau + \omega_j t)}{\omega - \omega_j} \right] \Big|_0^t = \frac{1}{2\omega_j} \left\{ \left[\frac{\sin(\omega t)}{\omega + \omega_j} - \frac{\sin(\omega t)}{\omega - \omega_j} \right] - \left[\frac{\sin(-\omega_j t)}{\omega + \omega_j} - \frac{\sin(\omega_j t)}{\omega - \omega_j} \right] \right\} \\ &= \frac{1}{2\omega_j} \left\{ \frac{-2\omega_j \sin(\omega t)}{\omega^2 - \omega_j^2} + \frac{2\omega_j \sin(\omega_j t)}{\omega^2 - \omega_j^2} \right\} = \frac{1}{\omega^2 - \omega_j^2} \left(\frac{\omega}{\omega_j} \sin(\omega_j t) - \sin(\omega t) \right) \end{aligned}$$

\Rightarrow

$$\ddot{q}_1 + q_1 = \sqrt{2/3} \sin(3t) \rightarrow \omega = 3, \omega_1 = 1$$

$$q_1 = \frac{1}{\omega_1} \int_0^t \sqrt{2/3} \sin(3\tau) \sin(\omega_1(t - \tau)) d\tau = \sqrt{2/3} \frac{1}{9-1} \left(\frac{3}{1} \sin(t) - \sin(3t) \right) = \frac{\sqrt{2/3}}{8} (3 \sin(t) - \sin(3t))$$

$$\ddot{q}_2 + 4q_2 = \sqrt{1/3} \sin(3t) \rightarrow \omega = 3, \omega_2 = 2$$

$$q_2 = \frac{1}{\omega_2} \int_0^t \sqrt{1/3} \sin(3\tau) \sin(\omega_2(t - \tau)) d\tau = \sqrt{1/3} \frac{1}{9-4} \left(\frac{3}{2} \sin(2t) - \sin(3t) \right) = \frac{\sqrt{1/3}}{5} \left(\frac{3}{2} \sin(t) - \sin(3t) \right)$$

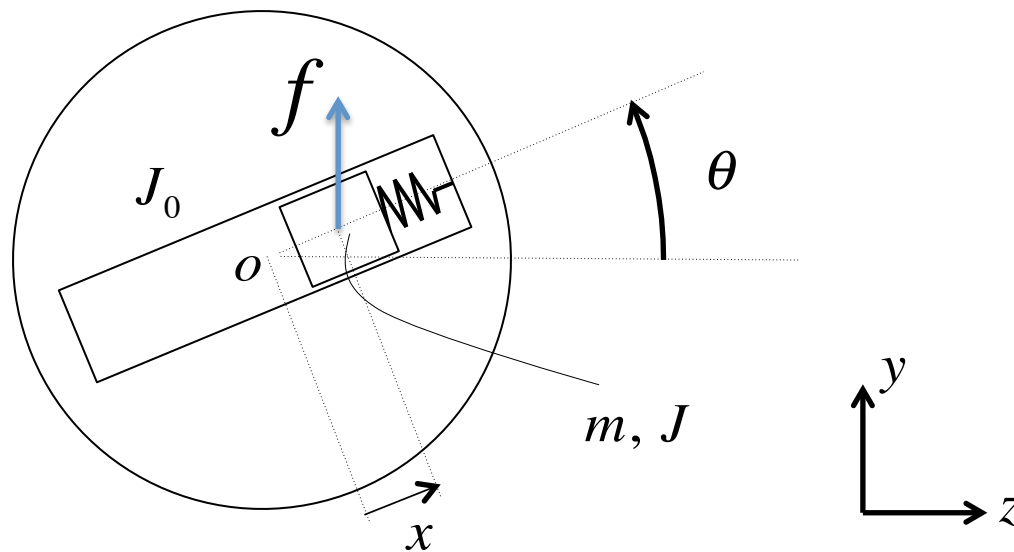
(f)

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = [X] \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \sqrt{2/3} & \sqrt{1/3} \\ \sqrt{1/3} & -\sqrt{2/3} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2/3}}{8} (3 \sin(t) - \sin(3t)) \\ \frac{\sqrt{1/3}}{5} \left(\frac{3}{2} \sin(t) - \sin(3t) \right) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2/3} \cdot \frac{\sqrt{2/3}}{8} (3 \sin(t) - \sin(3t)) + \sqrt{1/3} \cdot \frac{\sqrt{1/3}}{5} \left(\frac{3}{2} \sin(t) - \sin(3t) \right) \\ \sqrt{1/3} \cdot \frac{\sqrt{2/3}}{8} (3 \sin(t) - \sin(3t)) - \sqrt{2/3} \cdot \frac{\sqrt{1/3}}{5} \left(\frac{3}{2} \sin(t) - \sin(3t) \right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{20} \sin(t) - \frac{3}{20} \sin(3t) \\ \frac{\sqrt{2}}{40} \sin(t) + \frac{\sqrt{2}}{40} \sin(3t) \end{bmatrix}$$

(2) (40%)



See above. A round table has a slot, inside the slot is a block of mass m and rotary inertia J . The mass is coupled to the table by a spring of spring-constant k , and it slides along the slot without friction. The spring is fully relax when the block is at the center of the table. The round table has center of mass O and rotary inertia J_o with respect to O . External force f is applied to the block in the y direction. In response, the table rotates around O by angle θ and the block slides off-center by x .

- (a) Calculate the Kinetic energy T of the system.
- (b) Calculate the potential energy V of the system.
- (c) Using *Lagrange* formula with x and θ as the generalized displacements, drive the dynamics equations of the system when there is no external force (free vibration), .
- (d) Continue (c), drive the dynamics equations of the system when force f is applied (forced motion).

(a)

$$\begin{aligned} T &= \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} m \left[(x \cos \theta)' ^2 + (x \sin \theta)' ^2 \right] + \frac{1}{2} J \dot{\theta}^2 \\ &= \frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m \left[(\dot{x} \cos \theta - x \sin \theta \cdot \dot{\theta})^2 + (\dot{x} \sin \theta + x \cos \theta \cdot \dot{\theta})^2 \right] \\ &= \frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m (\dot{x}^2 \cos^2 \theta - 2x\dot{x}\dot{\theta} \sin \theta \cos \theta + x^2 \sin^2 \theta \cdot \dot{\theta}^2 + \dot{x}^2 \sin^2 \theta + 2x\dot{x}\dot{\theta} \sin \theta \cos \theta + x^2 \cos^2 \theta \cdot \dot{\theta}^2) \\ &= \frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m (\dot{x}^2 + x^2 \dot{\theta}^2) \end{aligned}$$

(b)

$$V = \frac{1}{2} kx^2$$

(c)

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = 0$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = 0$$

\Rightarrow

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{x}} \left(\frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m (\dot{x}^2 + x^2 \dot{\theta}^2) \right) \right] - \frac{\partial}{\partial x} \left(\frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m (\dot{x}^2 + x^2 \dot{\theta}^2) \right) + \frac{\partial}{\partial x} \left(\frac{1}{2} kx^2 \right) = 0$$

$$\frac{d}{dt} \left[\frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m (\dot{x}^2 + x^2 \dot{\theta}^2) \right) \right] - \frac{\partial}{\partial \theta} \left(\frac{1}{2} (J_0 + J) \dot{\theta}^2 + \frac{1}{2} m (\dot{x}^2 + x^2 \dot{\theta}^2) \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{2} kx^2 \right) = 0$$

\Rightarrow

$$m\ddot{x} - mx\dot{\theta}^2 + kx = 0$$

$$(J_0 + J + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} = 0$$

(d)
External force f causes a displacement of $x \sin \theta$ along the f 's direction

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{x}} \right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = f \cdot \frac{\partial (x \sin \theta)}{\partial x}$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\theta}} \right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = f \cdot \frac{\partial (x \sin \theta)}{\partial \theta}$$

\Rightarrow

$$m\ddot{x} - mx\dot{\theta}^2 + kx = f \cdot \sin \theta$$

$$(J_0 + J + mx^2)\ddot{\theta} + 2mx\dot{x}\dot{\theta} = f \cdot x \cos \theta$$