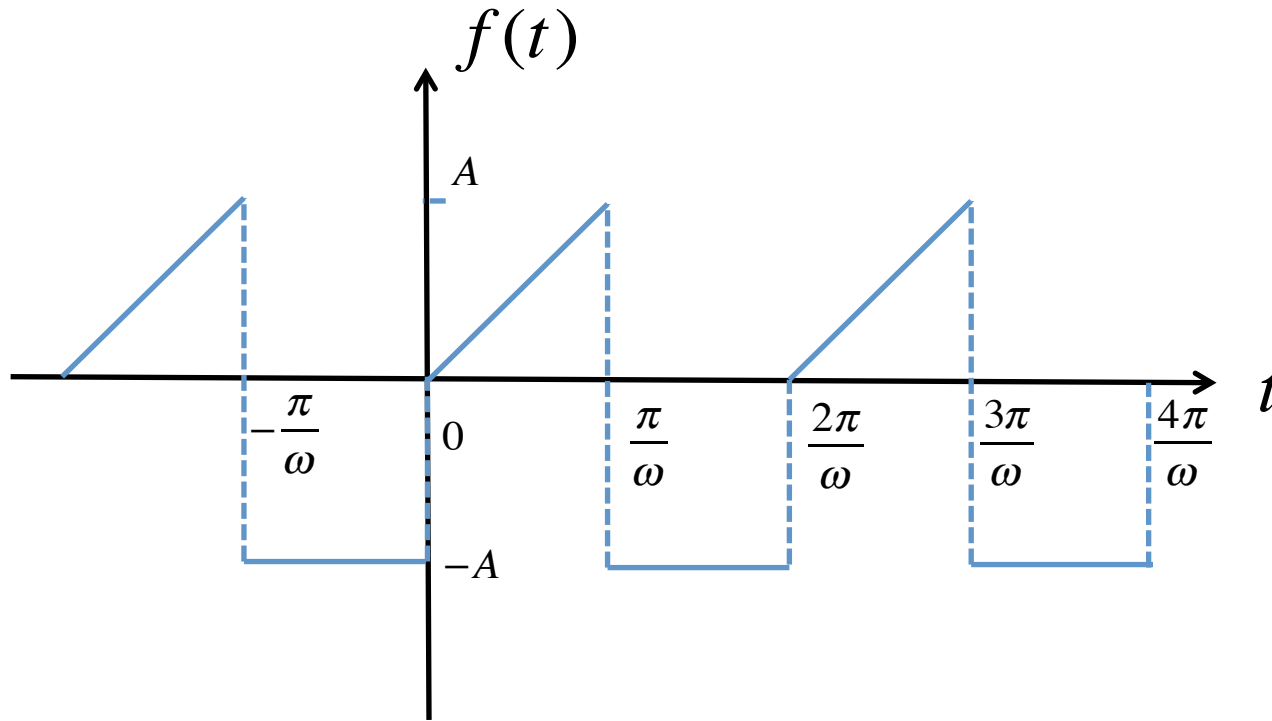


(1) Fourier-expand the following periodic input force:



$$\begin{aligned} f(t) &= \frac{a_0}{2} + a_1 \cos(\omega t) + a_2 \cos(2\omega t) + \dots \\ &\quad + b_1 \sin(\omega t) + b_2 \sin(2\omega t) + \dots \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \end{aligned}$$

Find the coefficients a_0 , a_n , and b_n

Solution

$$\begin{aligned}a_0 &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} f(t) dt \\&= \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} \frac{A\omega}{\pi} t dt + \frac{\omega}{\pi} \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (-A) dt = \frac{A\omega^2}{\pi^2} \int_0^{\frac{\pi}{\omega}} t dt - \frac{A\omega}{\pi} \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} 1 \cdot dt \\&= \frac{A\omega^2}{\pi^2} \left(\frac{1}{2} t^2 \right) \Big|_0^{\frac{\pi}{\omega}} - \frac{A\omega}{\pi} (t) \Big|_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} = \frac{A\omega^2}{2\pi^2} \left(\frac{\pi^2}{\omega^2} - 0 \right) - \frac{A\omega}{\pi} \left(\frac{2\pi}{\omega} - \frac{\pi}{\omega} \right) = \frac{A}{2} - A \\&= -\frac{A}{2}\end{aligned}$$

$$\begin{aligned}a_n &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} f(t) \cos(n\omega t) dt = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} \frac{A\omega}{\pi} t \cos(n\omega t) dt + \frac{\omega}{\pi} \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (-A) \cos(n\omega t) dt \\&= \frac{A\omega^2}{\pi^2} \int_0^{\frac{\pi}{\omega}} t \cos(n\omega t) dt - \frac{A\omega}{\pi} \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \cos(n\omega t) dt \\&= \frac{A\omega^2}{\pi^2} \left[\frac{1}{n\omega} t \sin(n\omega t) + \frac{1}{n^2 \omega^2} \cos(n\omega t) \right] \Big|_0^{\frac{\pi}{\omega}} - \frac{A\omega}{\pi} \left[\frac{1}{n\omega} \sin(n\omega t) \right] \Big|_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \\&= \frac{A\omega^2}{\pi^2} \left[\left(0 + \frac{1}{n^2 \omega^2} \cos(n\pi) \right) - \left(0 + \frac{1}{n^2 \omega^2} \cos(0) \right) \right] - \frac{A\omega}{\pi} [0 - 0] \\&= \frac{A\omega^2}{\pi^2} \left(\frac{1}{n^2 \omega^2} (-1)^n - \frac{1}{n^2 \omega^2} \right) \\&= \frac{A}{n^2 \pi^2} [(-1)^n - 1] \\&= \begin{cases} -\frac{2A}{n^2 \pi^2} & \text{when } n \in \text{odd} \\ 0 & \text{when } n \in \text{even} \end{cases}\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} f(t) \sin(n\omega t) dt = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} \frac{A\omega}{\pi} t \sin(n\omega t) dt + \frac{\omega}{\pi} \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (-A) \sin(n\omega t) dt \\
&= \frac{A\omega^2}{\pi^2} \int_0^{\frac{\pi}{\omega}} t \sin(n\omega t) dt - \frac{A\omega}{\pi} \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \sin(n\omega t) dt \\
&= \frac{A\omega^2}{\pi^2} \left[-\frac{1}{n\omega} t \cos(n\omega t) + \frac{1}{n^2\omega^2} \sin(n\omega t) \right] \Bigg|_0^{\frac{\pi}{\omega}} + \frac{A\omega}{\pi} \left[\frac{1}{n\omega} \cos(n\omega t) \right] \Bigg|_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} \\
&= \frac{A\omega^2}{\pi^2} \left[\left(-\frac{\pi}{n\omega^2} \cos(n\pi) + 0 \right) - (0 + 0) \right] + \frac{A}{n\pi} [\cos(2n\pi) - \cos(n\pi)] \\
&= -\frac{A}{n\pi} (-1)^n + \frac{A}{n\pi} [1 - (-1)^n] \\
&= \frac{A}{n\pi} [1 - 2(-1)^n] \\
&= \begin{cases} \frac{3A}{n\pi} & \text{when } n \in \text{odd} \\ -\frac{A}{n\pi} & \text{when } n \in \text{even} \end{cases}
\end{aligned}$$

(2) Given a standard mass-spring-damper mechanical system:

$$\ddot{x} + 2\dot{x} + 5x = f(t), \quad 0 < \zeta < 1$$

$f(t)$ is a periodic force with Fourier expansion:

$$f(t) = \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \cos(k\omega t)$$

Calculate the steady state response of x , which is x_p , the particular solution for $\ddot{x} + 2\dot{x} + 5x = f(t)$.

$$\text{Hint : } x_p = \sum_{k=0}^{\infty} x_{p,k} = \sum_{k=0}^{\infty} \alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t),$$

Calculate α_k & β_k

Solution

$$\ddot{x} + 2\dot{x} + 5x = f(t) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{k} \cos(k\omega t)$$

$$x_{p,k} = \alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t)$$

\Rightarrow

$$\dot{x}_{p,k} = -\alpha_k k\omega \sin(k\omega t) + \beta_k k\omega \cos(k\omega t)$$

$$\ddot{x}_{p,k} = -\alpha_k k^2 \omega^2 \cos(k\omega t) - \beta_k k^2 \omega^2 \sin(k\omega t)$$

\Rightarrow

$$-\alpha_k k^2 \omega^2 \cos(k\omega t) - \beta_k k^2 \omega^2 \sin(k\omega t) + 2[-\alpha_k k\omega \sin(k\omega t) + \beta_k k\omega \cos(k\omega t)] + 5[\alpha_k \cos(k\omega t) + \beta_k \sin(k\omega t)]$$

$$= \frac{2}{k\pi} \cos(k\omega t)$$

$$= (-k^2 \omega^2 \alpha_k + 2\beta_k k\omega + 5\alpha_k) \cos(k\omega t) + (-\beta_k k^2 \omega^2 - 2\alpha_k k\omega + 5\beta_k) \sin(k\omega t)$$

$$= [(5 - k^2 \omega^2) \alpha_k + 2k\omega \cdot \beta_k] \cos(k\omega t) + [-2k\omega \cdot \alpha_k + (5 - k^2 \omega^2) \cdot \beta_k] \sin(k\omega t)$$

\Rightarrow

$$\begin{cases} (5 - k^2 \omega^2) \alpha_k + 2k\omega \cdot \beta_k = \frac{2}{k\pi} \\ -2k\omega \cdot \alpha_k + (5 - k^2 \omega^2) \cdot \beta_k = 0 \end{cases}$$

\Rightarrow

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = \begin{bmatrix} 5 - k^2 \omega^2 & 2k\omega \\ -2k\omega & 5 - k^2 \omega^2 \end{bmatrix}^{-1} \begin{bmatrix} \frac{2}{k\pi} \\ 0 \end{bmatrix} = \frac{1}{(5 - k^2 \omega^2)^2 + (2k\omega)^2} \begin{bmatrix} 5 - k^2 \omega^2 & -2k\omega \\ 2k\omega & 5 - k^2 \omega^2 \end{bmatrix} \begin{bmatrix} \frac{2}{k\pi} \\ 0 \end{bmatrix}$$

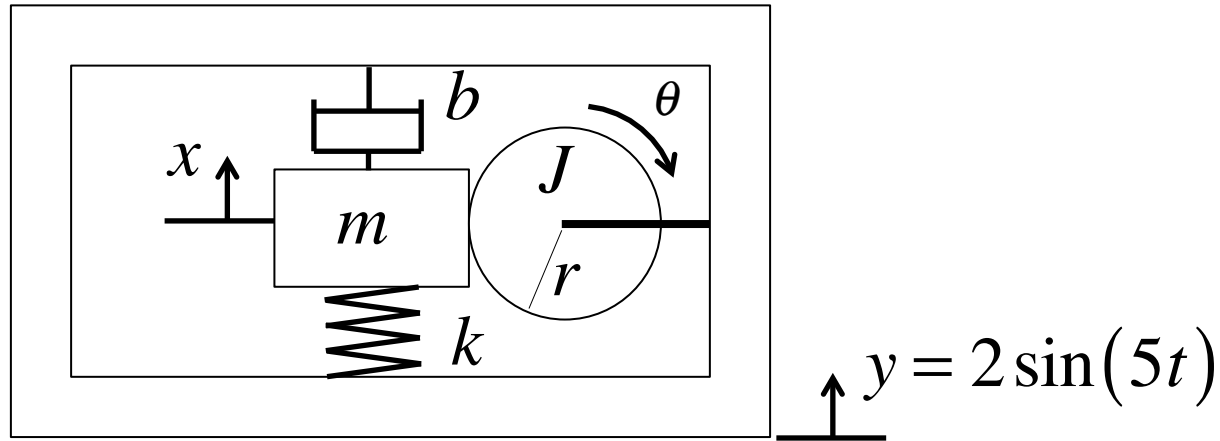
$$= \frac{1}{(5 - k^2 \omega^2)^2 + (2k\omega)^2} \begin{bmatrix} \frac{2}{k\pi} (5 - k^2 \omega^2) \\ \frac{4\omega}{\pi} \end{bmatrix}$$

\Rightarrow

$$\begin{aligned}x_p &= \sum_{k=0}^{\infty} \frac{1}{\left(5 - k^2 \omega^2\right)^2 + (2k\omega)^2} \left[\frac{2}{k\pi} (5 - k^2 \omega^2) \cos(k\omega t) + \frac{4\omega}{\pi} \sin(k\omega t) \right] \\&= \sum_{k=0}^{\infty} \frac{2}{k\pi} \cdot \frac{1}{\left(\omega_n^2 - k^2 \omega^2\right)^2 + (2k\omega)^2} \left[(5 - k^2 \omega^2) \cos(k\omega t) + 2k\omega \sin(k\omega t) \right]\end{aligned}$$

(3)

A block with mass m is suspended inside a box via a spring (k) and a damper (b). A roller with rotary inertia J and radius r is hinged to the box. The block and the roller are coupled by a set of gear and gear rack, such that $x = r\theta$. The box is shaken up and down following a function $y = 2 \sin(5t)$.



This system can be described by a standard mass-spring-damper-force format:

$$M\ddot{x} + B\dot{x} + Kx = F$$

Calculate M , B , K , and F .

Hint:

You can use either of the following methods to solve this problem:

(1) Calculate the equivalent mass with respect to displacement x .

(2) Assume there is a tangent force between the block and the roller, layout two dynamics equations using

$\sum F = m\ddot{x}$ and $\sum F \cdot r = J\ddot{\theta}$, and finally combine all the equations into one using $x = r\theta$.

Solution

$$m\ddot{x} = -k(x - y) - b(\dot{x} - \dot{y}) - T$$

$$J\ddot{\theta} = Tr$$

$$x = r\theta$$

\Rightarrow

$$T = J \frac{\ddot{\theta}}{r} = J \frac{\ddot{x}}{r^2}$$

$$m\ddot{x} = -k(x - y) - b(\dot{x} - \dot{y}) - \frac{J}{r^2} \ddot{x}$$

$$\left(m + \frac{J}{r^2}\right) \ddot{x} + b\dot{x} + kx = ky + b\dot{y}$$

$$y = 2 \sin(5t)$$

$$\dot{y} = 10 \cos(5t)$$

\Rightarrow

$$\left(m + \frac{J}{r^2}\right) \ddot{x} + b\dot{x} + kx = 2k \sin(5t) + 10b \cos(5t)$$

