Columbia University Department of Economics
S4415Q Game Theory, Problem Set 3

Due Monday August 1st in class.

Problem 1

Suppose two players play an infinitely repeated version of the stage game represented in the following payoff matrix:

1\2	A	B
A	2, 2	0, 0
B	0, 0	1, 1

(a) Consider the strategy profile in which both players play A regardless of past history. What is the lifetime payoff to each player for discount factor $\delta = 0.99$? $\delta = 0.5$? $\delta = 0.01$? For general δ?

(b) Is the strategy in part (a) a SPNE?

(c) Consider the strategy profile in which both players play B regardless of past history. What is the lifetime payoff to each player for $\delta = 0.99$, $\delta = 0.5$ and $\delta = 0.01$? For general δ?

(d) Is the strategy in part (c) a SPNE?

Problem 2

Consider the stage game represented in the following payoff matrix:

1\2	A	B
A	2, 2	0, 0
B	-2, -2	6, 6

(a) What are the pure-strategy Nash equilibria of this stage game?

Now suppose that the game is played twice, and both players have a discount factor of 1.

(b) Draw the extensive form of this game, making sure to indicate strategies, payoffs, and information sets.

(c) Is the strategy profile where both players play A at every information set a subgame perfect Nash equilibrium? Explain.

(d) Consider the following strategy profile:

Player 1: A in stage 1; B at all information sets in stage 2
Player 2: B in stage 1; B in stage 2 if Player 1 plays A and Player 2 plays B in stage 1, A in all other information sets in stage 2

Is this strategy profile a Nash equilibrium? Is it subgame perfect? Explain.

Now suppose that the game is infinitely repeated, and both players have a discount factor of 0.5. (Assume that the players are informed of their stage payoffs at the end of each stage.)

(e) Consider the strategy profile where both players play A, then B, then A, then B, etc., alternating *ad infinitum*[1]. Is this strategy profile a subgame perfect Nash equilibrium? Explain.

(f) What is the discounted sum of lifetime payoffs to each player from this strategy profile?

Problem 3

Suppose two players with discount factor δ play an infinitely repeated Prisoners Dilemma with the following stage game:

1\2	N	C
N	6, 6	-1,8
C	8, -1	0, 0

Consider the strategy in which the norm for both players is a 4 period cycle: They play (N,N) in the first period, (N,C) in the second period, (C,N) in the third period, (C,C) in the fourth, and then repeat that pattern over and over again. In the event that either player deviates from the norm (and doesn't follow the pattern), both players switch to playing (C,C) forever.

(a) Compute the lifetime payoff (call it v) to player 1 from the norm described above starting in the first period (with (N, N)). Show that it can be derived from:

$$v = [6 - \delta + 8\delta^2] + \delta^4 v.$$

(b) Compute the lifetime payoff to player 1 from the norm starting in the second, third and fourth periods.

(c) Suppose that during the first or second period, player 1 deviates and plays C instead of N. What are the lifetime payoffs from deviating starting in period 1? In period 2?

(d) Derive the conditions under which it is not profitable for player 1 to deviate in either the first or the second period[2].

(e) Is it ever profitable to deviate in the 3rd or 4th periods?

[1]Assume that they will continue to play this profile regardless of past history.
[2]Simplify the expression you find as much as you can, but you don't actually need to solve for an explicit δ.

Problem 4

Note that these questions are taken from CH17 of Dutta's *Strategies and Games* textbook. The setup is reworded here, but the model and parameters are identical.

There are two oil producers, Saudi Arabia (SA) and Venzuela (VA). Each can produce either *high* or *low* output. For SA, these output levels are $Q_H = 10$ mbd (million barrels per day) and $Q_L = 8$ mbd. For VA, the output levels are $q_H = 7$ mbd and $q_L = 5$ mbd. This means that total output can be either 13, 15, or 17 mbd.

Prices (and thus revenue) depend on whether demand is *good* or *bad*, and on total output (ie the combined output of both countries). Let p be the probability that demand is *good*. When demand is *bad*, prices (in terms of dollars per barrel) are \$16 when output is 13mbd, \$15 when output is 15mbd and \$14 when output is 17mbd. When demand is *good*, the prices are \$24 when output is 13mbd, \$21 when output is 15mbd, and \$18 when output is 17mbd.

Finally, there is a production cost for each firm of \$5 per barrel.

Thus, when demand is *bad*, the payoffs (ie profits, in terms of millions of dollars) are as follows:

SA\VA	q_L	q_H
Q_L	88, 55	80, 70
Q_H	100, 50	90, 63

(a) (17.1) Write down the payoff matrix for *good* demand periods.

(b) (17.9) Establish the two discount factor conditions that need to be satisfied in order for OPEC to be able to sustain the profit-maximizing output pattern of *low* output in *good* demand periods but *high* output in *bad* demand periods when the punishment used is the grim trigger punishment.[3]

(c) (17.10) How would your answer be any different if OPEC used the forgiving trigger instead - and chose to punish (ie overproduce) for T periods only? Explain.

(d) (17.11) If the punishment used is the forgiving trigger, is there any reason to punish SA's transgressions any differently than VA's - for example, by having punishment lengths be different in the two cases? Explain your answer carefully and be sure to do some computations.

(e) (17.12) Establish the exact discount rate requirements for each country for collusion sustainability if $p = \frac{1}{2}$.

(f) (17.13) In the last case and if $\delta = \frac{3}{4}$ for both SA and VA, how much would SA have to pay VA for the latter not to overproduce? Explain any assumptions that you make in computing your answer.

[3]Note that for parts b), c), d), f) your solutions may be functions of p.

Problem 5

Consider an infinitely-repeated version of the following stage game:

1\2	L	C	R
U	0, 0	2, $\underline{2}$	$\underline{4}$, 1
M	$\underline{2}$, $\underline{5}$	$\underline{5}$, 1	$\underline{4}$, 4
D	$\underline{2}$, $\underline{1}$	0, 0	2, $\underline{1}$

(a) Find all Nash equilibria in the stage game.

(b) What is Player 1's Min-Max Payoff? What is Player 2's?

(c) On a chart with π_1 on one axis, and π_2 on the other, plot each element of $\tilde{\pi}$, then clearly draw the set of feasible payoffs $\tilde{\Pi}$. (You don't need to make a mess by coloring in the feasible set - just draw the outline and label it).

(d) Plot each player's Min-Max Payoff on the chart.

(e) Clearly denote on your chart the set of average discounted payoffs that are guaranteed to be attainable by the Min-Max Folk Theorem. (Again, just the labelled outline is fine).

(f) For each of the following, write down whether or not the Min-Max Folk Theorem guarantees that it is attainable as the average discounted payoff in an SPNE: (5,4), (3,3), (2,1).