Final Exam Conjectures
MA 740
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Each of the following conjectures will appear, in one form or another, on the Final Exam. They will be worth a lot of points. Determine whether each one is, in general, true or false. Recall that a universal statement is true if it is true for all possible cases while it is false if there is even one counterexample. Be prepared to prove that your answer is correct by supplying a proof or counterexample, whichever one is appropriate.

All variables represent integers. p and q represent odd prime numbers.

1. Any prime of the form $5n + 1$ is also of the form $10n + 1$.

2. Any prime of the form $5n + 1$ is also of the form $15n + 1$.

3. The only prime of the form $n^3 - 1$ is 7.

4. There is a positive integer n such that $n/2$ is the square of an integer, $n/3$ is the cube of an integer, $n/5$ is the 5th power of an integer, and $n/7$ is the 7th power of an integer.

5. Suppose a, b, and m are integers with $m > 1$. If $a^3 \equiv b^3 \pmod{m}$ then $a \equiv b \pmod{m}$.

6. Suppose a, b, and m are integers with $m > 1$. If $ab \equiv 0 \pmod{m}$ then $a \equiv 0 \pmod{m}$ or $b \equiv 0 \pmod{m}$.

7. Let $n = pq$, with $p \neq q$. If $a^2 \equiv b^2 \pmod{n}$, $a \not\equiv b \pmod{n}$, and $a \not\equiv -b \pmod{n}$, then either $(a + b, n) = p$ or $(a + b, n) = q$.

8. Suppose a, b, and m are integers with $m > 1$. If $a^2 \equiv b^2 \pmod{m^2}$ then $a \equiv b \pmod{m}$ or $a \equiv -b \pmod{m}$.

9. Suppose that $m > 0$ and that $4m + 1, 8m + 1,$ and $12m + 1$ are prime. Then $n = (4m + 1)(8m + 1)(12m + 1)$ is a Carmichael number.

10. If n is a positive integer that is not square free then there are integers a and b such that $a \not\equiv b \pmod{n}$ but so that $a^k \equiv b^k \pmod{n}$ for all $k > 1$. (Note: An integer is square free if it is not divisible by the square of any integer larger than 1.)

11. If p is prime and $ord_p(a) = 4$, then $1 + a + a^2 + a^3 \equiv 0 \pmod{p}$

12. If $\left(\frac{a}{p}\right) = -1$ and $\left(\frac{b}{q}\right) = -1$ then $x^2 \equiv ab \pmod{pq}$ has a solution.

13. If $7 \nmid a$ then either $a^3 + 1$ or $a^3 - 1$ is divisible by 7.

14. If $m = 4^n + 1$ with $n > 0$ and m is prime then $3^{(m-1)/2} \equiv -1 \pmod{m}$.

15. Let p be a prime, If $ord_p(a) = h$ and h is even, then $a^{h/2} \equiv -1 \pmod{p}$.