Name______________________________ Date___________________________________

Chaos Theory Portfolio

Directions: In this portfolio, you will use repeated function composition to explore elementary ideas that are used in the mathematical field of chaos theory. Items under the Questions headings will be submitted to your teacher as part of your portfolio assessment. For all questions, make sure to be complete in your responses. This can include details such as the function being iterated, the initial values used, and the number of iterations. The phrase *many iterations* is used in some of the questions. Interpret that to mean using enough iterations so that you can come to a conclusion. If necessary, round decimals to the nearest ten-thousandth.

Introduction

In this unit, you learned how to use function operations. One of the most important operations is function composition. Just as two functions, *f* and *g*, can be composed with each other, a function, *f*, can be composed with itself. Everytime that a function is composed with itself, it is called an **iteration**. Iterations can be noted using a superscript. You can rewrite $(f \circ f)(x)$ as $f^2(x)$, $(f \circ f \circ f)(x)$ as $f^3(x)$, and so on. For this work, it is recommended that you use technology such as a graphing calculator.

Example 1

Start with the basic function $f(x) = 2x$. If you have an initial value of 1, then you end up with the following iterations.

- $f(1) = 2 \cdot 1 = 2$
- $f^2(1) = 2 \cdot 2 \cdot 1 = 4$
- $f^3(1) = 2 \cdot 2 \cdot 2 \cdot 1 = 8$

Questions

1. If you continue this pattern, what do you expect would happen to the numbers as the number of iterations grows? Check your result by conducting at least 10 iterations.
2. Repeat the process with an initial value of -1. What happens as the number of iterations grows?

Example 2

For this example, use the function $f(x)=\frac{1}{2}x+1$ and an initial value of 4. Note that with each successive iteration, you can use the previous output as your new input to the function.

- $f(4)=\frac{1}{2}\cdot 4+1=3$
- $f^2(4)=f(3)=\frac{1}{2}\cdot 3+1=2.5$
- $f^3(4)=f(2.5)=\frac{1}{2}\cdot 2.5+1=2.25$

Questions

3. What happens to the value of the function as the number of iterations increases?
4. Choose an initial value that is less than zero. What happens to the value of the function as the number of iterations increases?
5. Come up with a new linear function that has a slope that falls in the range $-1<m<0$. Choose two different initial values. For this new linear function, what happens to the function's values after many iterations? Are the function's values getting close to a particular number in each case?
6. Use the function $g(x)=-x+2$ with initial values of 4, 2, and 1. What happens after many iterations with all three initial values? How do the results of all three iterations relate to each other?

Example 3

Nonlinear functions can lead to some interesting results. Using the function $g(x)=-2|x-2|+4$ and the initial value of 1.5 leads to the following result after many iterations.

- $g(1.5)=-2|1.5-2|+4=3$
- $g^2(1.5)=g(3)=-2|3-2|+4=2$
- $g^3(1.5)=g(2)=-2|2-2|+4=4$
- $g^4(1.5)=g(4)=-2|4-2|+4=0$
- $g^5(1.5)=g(0)=-2|0-2|+4=0$

At this point, further iterations of the function will repeatedly obtain zero. You can conclude that repeated iterations of the function with an initial value of 1.5 will lead to a value of zero. For most values of this function, repeated iterations will lead to looping values. The next set of questions will address this concept.

Questions

7. What is the loop that forms after many iterations when the initial value of *g* is 1.7?
8. What is the loop that forms after many iterations when the initial value of *g* is 1.72?

Example 4

Repeated iterations will not always lead to a single number or a looping pattern. You can use the function $f(x) = 4x - x^2$ to demonstrate this.

Questions

9. Choose an initial value that is between zero and 4 and is not a whole number. Iterate it using the function, *f*, ten times. If necessary, you can round your results to the nearest ten-thousandth.
10. Choose a second initial value that is 0.01 greater than the initial value from question 9. Iterate it using the function, *f*, ten times. If necessary, you can round your results to the nearest ten-thousandth.
11. Is there a relationship between the ten values from question 9 and the ten values in question 10?

Chaos Theory

Chaos theory is the branch of mathematics that studies how small changes in inputs to functions can lead to vast changes in overall results. There is the famous idea that if a butterfly flaps its wings on one side of the world, it can lead to a hurricane on the other side of the world. This idea is part of chaos theory.

Questions

12. Use the Internet to conduct research on real-world applications of chaos theory. Some examples of search terms to use are *chaos theory*, the *butterfly effect*, and *fractal*. Write 2–3 paragraphs on how chaos theory is used in today's world in various fields.