This Logistic Function Problem Set will give you practice with a realistic logistic function. It is an INDIVIDUAL assignment. Cite all resources used. **You may ask your Math 140 instructor any question, but refrain from asking anyone else.**

If you violate the INDIVIDUAL nature of the assignment, document the unauthorized help you receive by name and by the nature of the help received. There will be an academic penalty depending on the extent of the unauthorized help, but you will not be guilty of plagiarism. Undocumented unauthorized help will be treated as plagiarism.

Purpose:

- To extend your skill in using an extremely important generalized exponential function called the **logistic function**
- To develop, algebraically, the two forms (exponential and logarithmic) of the logistic function .
- To recognize valid applications of the logistic function in scenarios of constrained growth.

You will want the use of Microsoft Mathematics (think of it as a cool graphing calculator) or other graphing tool (TI or Casio calculator or web-based applet, or Graphmatica), and the Equation Editor built into MS Word 2007 or 2010, or MathType. MS Excel (or equivalent) can also be used for graphing or other calculations.

The primary submission must be composed in Microsoft Word or any equivalent.

The rules are as follows:

Submission via email is due on Day Seven of Module/Week 6 (2359 Eastern Time Zone). It <u>must</u> be in one of the following formats:

.docx/.xlsx

.doc/.xls (Google Docs is okay, but keep Private—do not share them or borrow others)

*.pdf (if you convert your work into Adobe Acrobat format)

If you use Open Source office suites, be sure to convert your documents into one of the acceptable formats listed.

The asterisk is where you enter “LastName FirstName Logistic Problem Set” (no special characters. Things like #, ? and * screw up your submission)

This assignment is due by the end of Day Two in Module 6, and is weighted **<u>7.5%</u>** of the final grade.

Math background.

Pure exponential growth is not a real-world phenomenon (there are a finite number of atoms in the observable universe, and the exponential function is a continuous function), but it can be a fine model for short periods of time. How fast does an exponential function grow? We'll discover shortly.

Even graphing an exponential function is far more difficult than textbooks lead you to believe. Using a blackboard/whiteboard with horizontal units 1 cm apart, graphing the function $y = 2^x$, the point (100 cm, 2^{100} cm) is not feasible to plot on any normal scale.

Task 1: Determine how far 2^{100} cm is, to the nearest light year.

IF one uses a logarithmic scale on the y – axis, though, then the graph would be a line (semi-log paper)!

Nonetheless, if the growth is slow enough—on the order of a couple percent per year, for example, an exponential model is simple and very good at tracking populations over a twenty year interval, but not much longer. Such functions as $y = (1.025)^x$ grow slowly enough to be realistic growth models for x values to 30 or so. Calculate 1.025^{30} to see why this is so…..

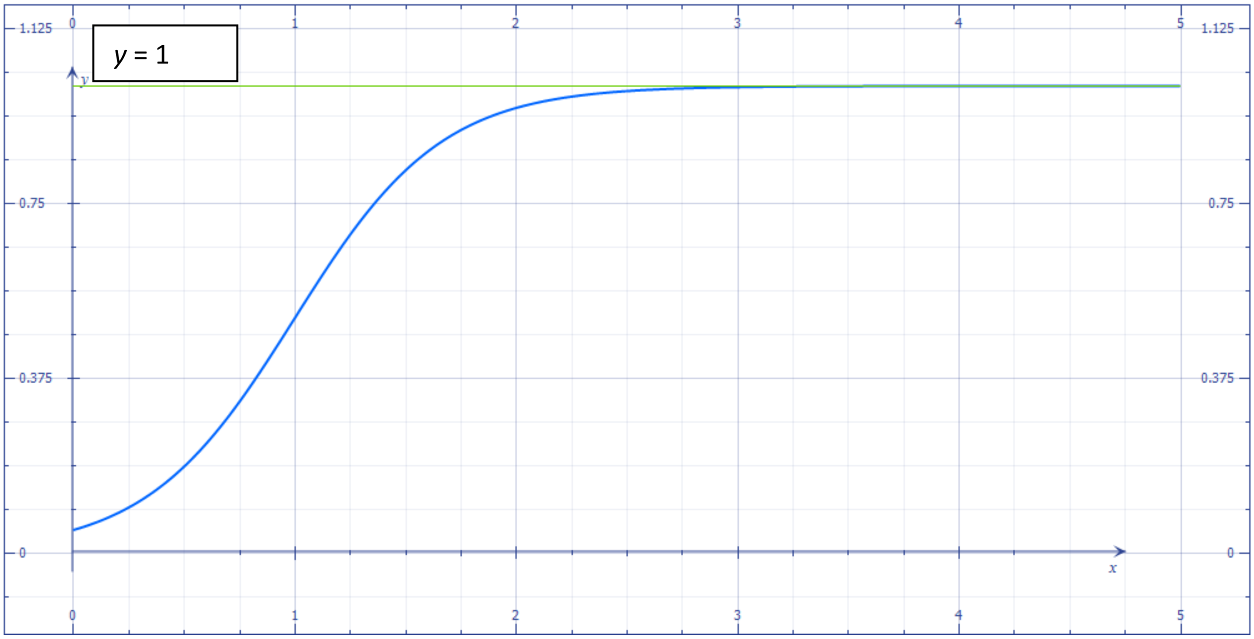
Logistic growth described

No living system exhibits exponential growth over any extended period of time, simply due to the exhaustion of available resources to feed the "beast". **It is this resource limitation (among other limiting factors) that causes an eventual "leveling" of the population (or size) of the system**.

With this in mind, think about **resource-constrained exponential growth**, also known as **logistic growth.** Google or Bing a guy named Verhulst (he lived in the 1800's) for background. It will be similar to the notes of this project.

Task 2: In a couple of paragraphs, describe and react to the issues Verhulst wrote about. Identify key famous people engaged in the debate of his day.

In order to model resource-constrained growth, you need a graph that looks more like the following instead of the pure exponential growth curve:



The upper limit of this graph was set at $y = 1$ to represent 100% of a system's capacity. As an organism's population or other resource-constrained variable increases, it exhibits almost-pure exponential growth early until it gets to about 50% of the environment's capacity, such as "amount of easily arable farmland brought to production", after which the pressures of limited supply slow the growth in easy-to-use farmland. Or in the case of an species, until it runs out of space or food supply.

The textbook provides you one of the many possible logistic growth models in Section 4.5, without explanation or interpretation. Here is its **exponential form**:

$$f(t) = \frac{c}{1 + ae^{-bt}} \quad (1)$$

In problems where 100% is the total capacity of a system (and for simplicity, one that won't change over time), we let c be the number 1. If the capacity increases, decreases, or oscillates over time (such as the seasons), then we'd replace c with whatever capacity function applies.

When we start measuring the system, we assign t the value of zero (also known as the initial condition of the system).

Note that, no matter what t is, since c is nonzero, then so is $f(t)$. Further, if you set $c = 1$, then $f(t)$ is the proportion of capacity the system achieves at time t.

Task 3:

a. In Equation 1, describe what $f(0)$ means, and calculate it for Equation 1.

b. Now let t get enormous. Over a very long period of time, what value does $f(t)$ approach in Equation 1?

The logarithmic form of the logistic growth function

In Chapter 4, you've learned that the logarithm can be used to "free" a variable from the exponent, if you will, for algebraic advantage. There are many good reasons to solve Equation 1 for t or b. Unfortunately, it takes a bit of algebra to get the equation into a form that would permit a simple rewrite into its logarithmic form. So, let's see what we can do algebraically to get e^{-bt} by itself:

Task 4: Have at it! Algebraically manipulate Equation 1 until e^{-bt} is by itself. That is, **solve for e^{-bt}.** But first, **replace $f(t)$ by y** to simplify the details.

Done right, you should get $e^{-bt} = \dfrac{c-y}{ay}$

Now, using the definition of logarithm in Section 4.2, you can rewrite this equation to solve for either b or t. For Equation (2), we'll solve for t:

$$-bt = \ln\left(\frac{c-y}{ay}\right)$$

$$t = \frac{-1}{b}\cdot\ln\left(\frac{c-y}{ay}\right)$$

$$t = \frac{-1}{b}\cdot\ln\left(\frac{c-f(t)}{af(t)}\right) \qquad (2)$$

Now, **you solve for *b* instead of *t*. *Call it Equation 3*.**

<u>Equations 2 and 3 are equivalent logarithmic forms</u> of the logistic equation (1).

One of the Big Ideas: If we know the population, or mass, or whatever quantity we are measuring at $t = 0$, we can use Equation (1) to quickly determine the value of a.

Once we know a, Equation 3 can be used to solve for b. And then finally, once we know a and b, we can find the time t it takes for a system to grow to whatever proportion of capacity we wish to predict using Equation 2.

The Application—Understanding the Spread of Influenza

Influenza (Google it) is a highly mutative virus that generally starts in livestock populations; eventually a strain develops that spreads to humans among the farmers of chickens and pigs in 3rd world countries not able to maintain strict standards of hygiene. New strains develop in the animal population fairly continuously, and the strongest of these varieties kill or greatly weaken their hosts. Symptoms manifest themselves more slowly than the disease spreads, hence its effectiveness. The World Health Organization has a regular strategy to "trap" new strains as they mutate before cross-species variants develop, and the CDC acts as coordinating agent in the US to gather information for the WHO. They also are the principal developers of flu vaccines (though in recent years, it had been outsourced to other nations to reduce costs. A move back to insourcing our own vaccine production began in 2012.)

Since the vast majority of people survive the flu, it isn't as critical to identify a new flu virus right away. In fact, a prospective new strain needs to show the ability to spread fairly rapidly and be rather potent before the government will spend large sums of money to build a vaccine for any new strain. H5N1 (called avian flu), however, showed itself to be rather deadly in Asia in the summer and fall of 2008, but it also proved to be tough to spread in the human population. It was H5N1's deadliness that led the Centers for Disease Control to set up a crash vaccine creation program for it in early 2009. By 2010, it was incorporated into the mainstream Fall season vaccine, along with two other strains showing early effectiveness in the 2009 season.

Vaccinations: The chief effect of a vaccine is to reduce the susceptible population size. If 50,000,000 people receive a vaccine and if it is timely and effective, then 50,000,000 people can be considered to be safe from the disease (or even to have "caught" it already). Using the logistic function as a model, this 16% reduction in the vulnerable population should also reduce the number infected by 16% at any given time.

Vaccinations also make a disease like the flu harder to spread, because clusters of vulnerable people in close quarters are fewer. This would reduce the *infectivity* of a virus, so that its spread becomes even slower, until the flu season passes and the virus goes largely dormant. Out-of-season flu epidemics are pretty rare. (None of the Tasks involve the effects of vaccination)

For the rest of the Problem Set, in Equation 1 we'll set $c = 1$. It will be your job to find the values for a and b under three separate Scenarios, explain what they represent for each scenario in Equations 1, 2, and 3, and then find the time it takes for a given set of proportions of the vulnerable population to be infected by it.

For U.S. population studies, where at least 250,000,000 people are vulnerable to a particular flu virus, if we set the capacity of the flu virus to infect vulnerable people in the US at $c = 1$, the value of the initial proportion $f(0)$ is considerably smaller than 1, (as few as 300 people with flu-like symptoms may have their blood tested by the time a new virus is identified) and so we'd set

$f(0) = 300/250{,}000{,}000$, a REALLY small decimal.

Okay, back to the project tasks.

You will evaluate the effects of changing $f(0)$ and changing b on the graphs of your logistic equations. **The value of *b* will be given as a proportion/day** so that if $b = .01$, then 1% of the vulnerable population is newly infected per day. Your value of $f(0)$ must be computed from the initial number of cases as a proportion of the 250,000,000 vulnerable. $f(0)$ will be a very small decimal number in most of the tasks. We'll call Day Zero (that is, you set $t = 0$ on Day Zero) the day when scientists have identified a new flu virus strain.

For all scenarios (except the one in Task 7), find a, and then express the logistic equation using Equation 1 as the template using the values for a and b, with $c = 1$.

Plot a graph for each scenario, ideally using MS Math 4.0 (free), or Graphmatica 2.0 (shareware) or even Excel if you already have the skills for plotting graphs in it. A Graphing calculator really won't be good enough for the plots—they're pretty primitive.

Scenario 1: Let's assume that $b = .0075$ and that on Day Zero, there are an estimated 10,000 infected people out of a vulnerable population of 250,000,000.

Scenario 2. Another strain of the flu is more virulent, with double the value of b as in Scenario 1. Let's also assume 10,000 people have been infected by Day Zero (same vulnerable population.

Scenario 3. Strain 3 is in its second year in the US, so it is estimated that 500,000 people have had that variant. Let us also assume that it was slightly less virulent than the second strain, so that $k = .008$.

Note: You may use Equations (1), (2) *or* (3) or some combination of *them to complete the following tasks. You may even graph Equation (1) and use a Trace facility to approximate the exact solutions.*

Task 5: For each of the Scenarios above, determine when half the population ($f(t) = 0.5$) has been infected, and estimate when 80% of the vulnerable population has "caught" that strain. Remember, your units for t are in "days", though that number may be large.

Task 6: Using the three Scenarios, compare the impact of the different b values, and compare the impact of the different $f(0)$ values. Which factor seems more important—the size of the initial population, or the virulence?

In this last task, your job is to find b, and you are given a different set of assumptions.

Task 7: Assume you start with an initial population of 100,000 infectees. Calculate what value of b will result in 60% of the population being infected as of Day 300.