Math 5420
Project ideas

1. **Quaternion group**. We know that we can expand the real line by defining the square root of -1, and there is a clear relationship between the complex numbers $\mathbb{C}$ and the 2-dimensional plane $\mathbb{R}^2$. Can we develop an analogous result for higher dimensions – that is to say, can we further expand $\mathbb{C}$? In 1843, William Hamilton answered this question by developing a system of quadruples (a, b, c, d) that satisfied rules of addition, scalar multiplication, and quaternion multiplication. Show what these operations are, define the structure they create, and show how they can be defined using matrices (example 3.3.7 p. 122 in Beachy and Blair). You might include a table for the quaterion group Q_8, some historical notes about Hamilton, and some words about the applications of quaternions in physics, computer graphics, etc.

2. **Rubik's cube group**. Discuss solving a Rubik's cube in terms of group theory. Can you describe the Rubik's cube group in terms of simpler, smaller groups?

3. **Cryptography**. Explain the difference between private-key and public-key cryptography, show how the RSA cryptosystem works and how it is related to factoring of large numbers, and do some examples. A good resource is chapter 7 of the book *Abstract Algebra, Theory and Applications* by Thomas W. Judson, available free online.

4. **Wallpaper groups**. Wallpaper groups are two-dimensional symmetry groups. They categorize patterns by their symmetries, and can be described as isometries of the Euclidean plane that contain two linearly independent translations. There are 17 possible wallpaper groups. Describe how these are defined, give examples, and describe some properties. A good resource is chapter 12 of the book *Abstract Algebra, Theory and Applications* by Thomas W. Judson.

5. **Lorentz group**. Start with Problem 18, p. 124 of Beachy and Blair which defines the Lorentz group and asks you to prove it is a subgroup of $GL_2(\mathbb{R})$. Then explain how this is related to Special Relativity. You might include an explanation of the differences between Newtonian dynamics and Special Relativity, and some history about the development of Special Relativity.

6. **Orbit-Stabilizer Theorem**. Problems 19-22 on page 114 of Beachy and Blair define the sets $C(a)$ and $Z(G)$ and give some examples using S_3 and $GL_2(\mathbb{R})$. You have done 19 and 21, so do the others. In addition to finding the centralizer of $(1, 2, 3)$ in S_3, also find that set in S_4, and in A_4. Centers and centralizers are related to group actions. Define what a group action is and and give some examples. You can use Beachy and Blair or Judson as a reference. An important and fairly accesible result in this area is the Orbit-Stabilizer Theorem, state this theorem and give an explanation of the proof.

If there are other topics you would prefer to do, please see me or send me an email.