1. (10 points) Suppose that A is an $n \times n$ matrix. We gave at least 10 statements that are equivalent to the statement "A is invertible." List 5 of these conditions. One of your conditions must be a criterion for invertibility that involves the eigenvalues of A.

2. Consider the data points $(0, 0),\ (1, 1),\ (2, 3)$.

   (a) (5 points) Find the line $y = c_0 + c_1 x$ which best fits the data in the least-squares sense.

(b) (5 points) Find a quadratic polynomial of the form $y = c_0 + c_1 x + c_2 x^2$ which passes through all three of the data points.

3. (10 points) Let

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}.$$

Compute a formula for A^k, where k is a positive integer. Your "answer" should be a single matrix.

4. Suppose that A is an $n \times n$ matrix with eigenvalue λ and corresponding eigenvector $\mathbf{v}$.

   (a) (3 points) If A is invertible, is $\mathbf{v}$ an eigenvector of A^{-1}? If so, what is the corresponding eigenvalue? If not, explain why not.

   (b) (3 points) Is $3\mathbf{v}$ an eigenvector of A? If so, what is the corresponding eigenvalue? If not, explain why not.

5. (5 points) Let

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Find the eigenvalues of A and their algebraic multiplicities.

6. Suppose that $\det(A) = 7$, where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Evaluate each of the following:

(a) (5 points) $\det\left(\begin{bmatrix} a+2d & b+2e & c+2f \\ 3g & 3h & 3i \\ d & e & f \end{bmatrix}\right)$

(b) (5 points) $\det\left(\begin{bmatrix} 2a & 2b & 2c \\ 3d-a & 3e-b & 3f-c \\ 2g & 2h & 2i \end{bmatrix}\right)$

7. Suppose that A is a diagonalizable matrix with characteristic polynomial

$$f_A(\lambda) = \lambda^2(\lambda - 3)(\lambda + 2)^3(\lambda - 4)^3.$$

(a) (2 points) Find the size of the matrix A.

(b) (4 points) Find the dimension of E_4, the eigenspace corresponding to the eigenvalue $\lambda = 4$.

(c) (4 points) Find the dimension of the kernel (nullspace) of A.

8. (7 points) Suppose that A is an $m \times n$ matrix such that $\ker(A) = \mathbf{0}$. Prove that $A^T A$ is invertible. Be sure to justify each step in your proof completely.

9. (5 points) Find, with proof, the possible values of the determinant of an $n \times n$ orthogonal matrix A.

10. (5 points) Suppose that A is an invertible matrix. Prove that

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

11. (7 points) Suppose that A is a diagonalizable $n \times n$ matrix and has only 1 and -1 as eigenvalues. Show that $A^2 = I_n$, where I_n is the $n \times n$ identity matrix.

12. (5 points) Find, with proof, all possible real eigenvalues of an orthogonal matrix.

13. (10 points) Determine whether each of the following statements is true or false. No justification is necessary. Each question is worth 1 point. You will earn 1 point for each question answered correctly, 0 points for each unanswered question, and -1 point for each question answered incorrectly, with a minimum possible score of 0 and a maximum possible score of 10.

(a) If A and B are symmetric $n \times n$ matrices, then AB must be symmetric as well.

(b) If A and S are orthogonal matrices, then $S^{-1}AS$ is also orthogonal.

(c) Suppose that V is a subspace of $\mathbb{R}^n$ with an orthonormal basis $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Let A be the matrix whose column vectors are the vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$. Then the matrix of the orthogonal projection onto V is AA^T.

(d) If an $n \times n$ matrix A does not have n distinct eigenvalues, then A is not diagonalizable.

(e) If the determinant of a 4×4 matrix A is 4, then $\text{rank}(A) = 4$.

(f) If A is an $n \times n$ matrix, then the determinant of A is equal to the product of its diagonal entries.

(g) The matrix $\begin{bmatrix} 3 & 1 & 1 & 1 & 4 \\ 1 & 3 & 1 & 1 & 1 \\ 1 & 1 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 & 1 \\ 4 & 1 & 1 & 1 & 3 \end{bmatrix}$ is orthogonally diagonalizable.

(h) If A and B are similar matrices, then $\det(A) = \det(B)$.

(i) If an $n \times n$ matrix A is diagonalizable, then there is a unique diagonal matrix that is similar to A.

(j) If 3 is an eigenvalue of an $n \times n$ matrix A, then 9 must be an eigenvalue of A^2.