1. (5 points) Solve the following system of equations:

$$\begin{aligned} x - y + z + w &= 5 \\ y - z + 2w &= 8 \\ 2x - y - 3z + 4w &= 18. \end{aligned}$$

2. (4 points) Find a basis for (a) the kernel (nullspace) and (b) the image (column space) of the matrix A given below:

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 4 \\ 1 & 2 & 1 & 1 & 6 \\ 0 & 1 & 1 & 1 & 3 \\ 2 & 2 & 0 & 1 & 7 \end{bmatrix}.$$

You may use the fact that

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix}.$$

Find the standard matrix representation of T, i.e. the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

4. (5 points) Suppose that A and B are $n \times n$ matrices such that A is similar to B. Show that A^k is similar to B^k for all integers $k \geq 1$.

5. (5 points) Suppose that A and B are 5×5 matrices such that $A\mathbf{x} \neq B\mathbf{x}$ for all non-zero vectors $\mathbf{x}$ in $\mathbb{R}^5$. Find the rank of the matrix $(A - B)$.

6. (5 points) Suppose that T is a linear transformation with standard matrix representation A, and that A is a 7×6 matrix such that the kernel of A has dimension 4. Find the dimension of the image of T.

7. (5 points) Suppose that A is a 4×4 matrix, B is a 4×3 matrix, and C is a 3×4 matrix such that $A = BC$. Show that A is not invertible.

8. (16 points) Answer each of the following questions **True** or **False**. No explanation is necessary. Each question is worth 2 points. You will receive 2 points for each question answered correctly, 0 points for each unanswered question, and -2 points for each question answered incorrectly, with a minimum possible score of 0.

   (a) If A is a 7×4 matrix, and the dimension of the image of A is 3, then the columns of A are linearly dependent.

   (b) Any six vectors in $\mathbb{R}^4$ must span $\mathbb{R}^4$.

   (c) If A is a matrix such that the linear system $A\mathbf{x} = \mathbf{0}$ has the unique solution $\mathbf{x} = \mathbf{0}$, then A must be a square matrix.

   (d) If A and B are 2×2 matrices such that $AB = 0$, then $BA = 0$.

(e) There exists a 3×2 matrix A such that $\ker(A) = \operatorname{im}(A)$.

(f) There exists a 3×3 matrix A such that $\ker(A) = \operatorname{im}(A)$.

(g) If A is an $n \times m$ matrix, then the set of vectors $\mathbf{x}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{R}^m$.

(h) If A is an $n \times m$ matrix, and $\mathbf{b}$ is an arbitrary vector in $\mathbb{R}^n$, then the set of vectors $\mathbf{x}$ in $\mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^m$.