Assignment 1: Due May 27

Please email your solutions (as a pdf file) to the TA Sujay Bhatt at
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Make sure your email contains your name and student number.

Notation: Below x' denotes transpose of a matrix or vector x.

1. Define what is meant by a permutation matrix. What is their purpose? Show that the product of permutation matrices is a permutation matrix.
   Show that the inverse of a permutation matrix is the transpose of the matrix.
2. Markowitz Portfolio Optimization: This exercise requires some elementary optimization (from your calculus course) together with basic linear algebra. Consider the following optimization problem:
$$\min_{x \in \mathbb{R}^n} x'Cx$$
$$\text{such that } g'x = r$$
   Here x is a portfolio allocation vector, C is an $n \times n$ covariance matrix, g denotes the rate of returns of individual components and the scalar r denotes the return of the entire portfolio. It is assumed that g, r, C are known. Find the optimal x.
3. Consider the equation $Ax = 0$ where A is an $m \times n$ matrix and 0 denotes the vector with elements 0. For the cases $m < n$, $m = n$ and $m > n$, discuss the solutions of this equation. (Disregard the trivial solution $x = 0$).
4. In abstract terms, explain how the LU decomposition of a matrix works. That is: starting with a matrix A and applying successive row-wise transformations using lower triangular matrices $L_1, L_2, \ldots$ show that one ends up with upper triangular matrix U. This abstraction is at the core of solving linear systems of equations by Gaussian elimination.
5. Is a linear system of equations well posed? This is crucial - because otherwise you can get nonsensical answers.
   Consider solving the system $Ax = b$ where A is an $m \times n$ matrix. Show that:
   (a) Either a solution x exists
   (b) OR there exists a vector y such that $A'y = 0$, $b'y \neq 0$.
   That is either (a) holds or (b) holds; but both (a) and (b) cannot hold.
   A more general statement (which arises in linear programming optimization) is: Consider solving the system $Ax = b$ where A is an $m \times n$ matrix and x is a non-negative vector. Show that:

(a) Either a solution x exists

(b) OR there exists a vector y such that $A'y \geq 0$, $b'y < 0$.

That is either (a) holds or (b) holds; but both (a) and (b) cannot hold.

6. This exercise is meant to give you some familiarity of pivoting and the so called echelon form of matrices. Compute the rank of the following matrices (justify your method)

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 6 & 2 & 4 \\ 2 & 0 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

7. The equation of an n-dimensional plane is $c'x = a$, where c is the normal to the plane and a is a scalar. Determine the point in the plane nearest the origin and find the distance of this point to the origin.
8. What is the equation of a line in n-dimensional space?
9. Show that the following matrix rotates vectors counter-clockwise (for example in computer graphics)

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$