Brief review of the Riemann integral (7.1-7.4 Goldberg)

A partition σ of $[a,b]$ is a finite set of points
$\sigma = \{x_0, x_1, \dots, x_n\}$ where
$a = x_0 < x_1 < \dots < x_n = b$.

If $f : [a,b] \to \mathbb{R}$ is bounded, and σ is a partition of $[a,b]$, define the upper sum $U(f,\sigma) = \sum_{i=1}^{n} \left[\sup_{x \in [x_{i-1}, x_i]} f(x)\right](x_i - x_{i-1})$,

and the lower sum $L(f,\sigma) = \sum_{i=1}^{n} \left[\inf_{x \in [x_{i-1}, x_i]} f(x)\right](x_i - x_{i-1})$.

Define $\overline{\int_a^b} f = \inf_{\sigma} U(f,\sigma)$, $\underline{\int_a^b} f = \sup_{\sigma} L(f,\sigma)$.

If $\overline{\int_a^b} f = \underline{\int_a^b} f$, we say f has a Riemann integral ($\int_a^b f$)

over $[a,b]$ and $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$.

Tools of proof use algebra relations related to refinements.

σ^*, σ are partitions of $[a,b]$. σ^* <u>refines</u> σ if

$\sigma^* \supseteq \sigma$.

Algebra properties: Assume σ^* refines σ. Then

$$L(f,\sigma) \le L(f,\sigma^*) \le U(f,\sigma^*) \le U(f,\sigma).$$

Primary tool to show existence of the Riemann integral.

Thm. Let $f : [a,b] \to \mathbb{R}$ be bounded. $\int_a^b f$ exists if and only if for each $\varepsilon > 0$ there exists a partition σ of $[a,b]$ such that

$$U(f,\sigma) - L(f,\sigma) < \varepsilon.$$

Important results.

Thm. If $f : [a,b] \to \mathbb{R}$ is continuous, then $\int_a^b f$ exists.

I recommend that you be able to prove this theorem. The Proof of the theorem is outlined in #6 p 184 Goldberg. It is an important application of f is continuous on a compact set implies f is uniformly continuous on a compact set.

Thm. $f : [a,b] \to \mathbb{R}$. $\int_a^b f$ exists if and only if the set of discontinuities of f on $[a,b]$ has measure 0.

We will define measure 0 and prove this result soon. I state it here because it is a fundamental and well know result and it can be proved without introducing Lebesgue measure. We outlined the proof in the MTH 330 notes.

Properties:

1. If $\int_a^b f$ exists, $c \in (a,b)$, then $\int_a^c f$, $\int_c^b f$ exist and

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

2. If $\int_a^b f$, $\int_a^b g$ exist, $\lambda_1, \lambda_2 \in \mathbb{R}$ then $\int_a^b \lambda_1 f + \lambda_2 g$ exists and

$$\int_a^b \lambda_1 f + \lambda_2 g = \lambda_1 \int_a^b f + \lambda_2 \int_a^b g .$$

[Seems harmless, but define $T: C[a,b] \to C[a,b]$, ~~that~~ by

$$Tf = \int_a^b f.$$ Then T is a linear operator.

You know from say MTH 310 or 565 that if T is a linear map from an m-dimensional vector space to an n-dimensional vector space, then T is represented by an $n \times m$ matrix. We don't know how to represent $\int_a^b f$ by a matrix which implies $C[a,b]$ is not finite dimensional. So, it seems harmless, but it is interesting.)

3. If $\int_a^b f$, $\int_a^b g$ exist, $f(x) \le g(x)$ on $[a,b]$ then
$$\int_a^b f \le \int_a^b g$$

(Again, this seems harmless, but this says that we can produce definitions such that the operator $Tf = \int_a^b f$ is an <u>increasing</u> operator.)

4. If $\int_a^b f$ exists, then $\int_a^b |f|$ exists and

$$\left| \int_a^b f \right| \le \int_a^b |f| .$$

Motivation for developing other types of integration.

Famous example. The Dirichlet function is not Riemann integrable.

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational,} \end{cases} \qquad 0 \le x \le 1.$$

$$\overline{S}_0^1 f = \inf_\sigma \sum_i (\overset{1}{\sup f}\ \cancel{\overset{0}{\inf f}})(x_i - x_{i-1}) = \inf_\sigma \sum_i (x_i - x_{i-1}) = \inf_\sigma (1-0) = 1$$

$$\underline{S}_0^1 f = \sup_\sigma \sum_i (\overset{0}{\inf f})(x_i - x_{i-1}) = \sup_\sigma \sum_i 0 = 0.$$

We will study the Lebesgues integral. There are multiple well-studied definitions of integration that have meaningful applications.

Riemann-Stieltjes integral: $\int_a^b f\,dg$

In MTH 168, we write $\int_a^b f\,dx$, so somehow $x = g(x)$.
Write the form
$$\int_a^b f\,g'(x)\,dx = \int_a^b f\,dg \quad \text{using differentials } dg = g'(x)\,dx.$$

So, the Riemann integral is on the line $y = x$ and the R-S integral relaxes this so you can integrate on a more general "surface" $y = g(x)$.

We do not study the R-S integral in these notes.
You are invited to study the R-S integral for a MTH 480 topic.

Defn $S \subseteq \mathbb{R}$ is said to be a set of <u>measure zero</u> if for each $\varepsilon > 0$ there exist open intervals $I_1, I_2, \dots$ (implied a countable collection) such that $\sum_{i=1}^{\infty} \text{length}(I_i) < \varepsilon$ and $\bigcup_{i=1}^{\infty} I_i \supseteq S$.

Facts

i) If A has measure 0, $B \subseteq A$, then B has measure 0.

$$\bigcup_{i=1}^{\infty} I_i \supseteq A \supseteq B$$

ii) If each of $A_1, A_2, \dots$ (countable collection) have ~~finite~~ measure 0, then $\bigcup_{i=1}^{\infty} A_i$ has measure 0.

Cover each S_k with open intervals $I_{1k}, I_{2k}, \dots$ and $\sum_{i=1}^{\infty} \text{length}(I_{ik}) < \varepsilon/2^k$.

iii) ϕ (the empty set) has measure 0.

Note: ii) implies a countable set has measure 0, since a point has measure 0.

There are examples of uncountable sets having measure 0 (the Cantor set)

<u>Defn</u> (open sets in relative topologies)

A set O is open in $[a,b]$ if $O = \hat{O} \cap [a,b]$ where $\hat{O}$ is open in $\mathbb{R}$.

So we will talk about open subintervals of $[a,b]$, $c,d \in [a,b]$

(c,d), $[a,d)$, $(c,b]$, $[a,b]$, ϕ.

Defn oscillation of f over I

$f: [a,b] \to \mathbb{R}$. Assume f is bounded. Let $I \subseteq [a,b]$ be an interval. Define the oscillation of f over I by

$$\omega_f(I) = \sup_I f - \inf_I f.$$

If $x \in [a,b]$ define $\omega_f(x) = \inf\{\omega_f(I) : I \ni \{x\},\ I \text{ an open subinterval of } [a,b]\}$. So $\omega_f : [a,b] \to [0,\infty)$.

Lemma (Thm 5.6 p 143 Goldberg). f is continuous at $x \iff \delta_f(x) = 0$

Outline of Proof.

$\Longleftarrow$ Let $x \in (a,b)$. Find an open subinterval of $[a,b]$ I such that $x \in I$ and $\sup_I f - \inf_I f < \varepsilon$. Note: we can choose, $a < c < x < d < b$. Let $\delta = \min\{x-c, d-x\}$. Assume $|y-x| < \delta$. Then $y \in I$ and $|f(y) - f(x)| \leq \sup_I f - \inf_I f < \varepsilon$.

If $x = a$, the Left endpt, then this is where the concept of open subinterval comes into play. Now $I = [a,d)$ and $\delta = d-a$.

$\Longrightarrow$ Let $x \in (a,b)$. Let $\varepsilon > 0$ Find $\delta > 0$ such that $a < x-\delta < x+\delta < b$ and if $|y-x| < \delta$ then $|f(y) - f(x)| < \frac{\varepsilon}{2}$. Let $I = (x-\delta, x+\delta)$. Then $|f(y) - f(x)| < \frac{\varepsilon}{2} \Rightarrow f(y) - \inf_I f \leq \frac{\varepsilon}{2} \Rightarrow \sup_I f - \inf f \leq \frac{\varepsilon}{2} < \varepsilon$.

Lemma: Let $\varepsilon > 0$. $A = \{x \in [a,b] : \omega_f(x) \geq \varepsilon\}$ is closed in $[a,b]$ (meaning) A^c is open in the relative topology on $[a,b]$.

Pf. Show A^c is open. $A^c = \{x \in [a,b] : \omega_f(x) < \varepsilon\}$. Find an open subinterval I of $[a,b]$ (so you treat $x=a$, $x=b$ separately) such that $x \in I$ and $I \subseteq A^c$.

So, $\omega_f(x) < \varepsilon$, so there exists $I \subseteq [a,b]$ such that $\omega_f(I) < \varepsilon$.

So if $y \in I$, $\omega_f(y) \leq \omega_f(I) < \varepsilon$ and $y \in A^c$.

Theorem. Let the Riemann integral $\int_a^b f$ exist. Then f is continuous almost everywhere on $[a,b]$ (write continuous a.e. on $[a,b]$) which means $\{x \in [a,b] : f \text{ is not continuous}\}$ has measure 0.

Proof. $E = \{x \in [a,b] : \omega_f(x) > 0\}$, the set of discontinuities of f. The proof is complete when we show E has measure 0.

For $m = 1, 2, \ldots$, define $E_m = \{x : \omega_f(x) \geq 1/m\}$.

Then $E = \bigcup_{m=1}^{\infty} E_m$ (~~Always~~, $E \supseteq E_m$ for each m so $E \supseteq \bigcup_{m=1}^{\infty} E_m$

For you, show $E \subseteq \bigcup_{m=1}^{\infty} E_m$.)

We show each E_m has measure 0. Then the countable union of sets of measure 0 has measure 0 and we are done.

To show E_m has measure 0: Fix m. Let $\varepsilon > 0$. Find a partition P such that $U(P, f) - L(P, f) < \varepsilon/m$. ($P$ exists because we assume $\int_a^b f$ exists.)

$P = \{a = x_0, x_1, \ldots, x_k = b\}$. Let $I_j = [x_j - x_{j-1}]$, $j = 1, \ldots, k$.

Then $\sum_{j=1}^{k} (\sup_{I_j} f - \inf_{I_j} f)(x_j - x_{j-1}) < \varepsilon/m$ or

$\sum_{j=1}^{k} \omega_f(I_j)(x_j - x_{j-1}) < \varepsilon/m$. (*)

Select those I_j's, say $I_{t_1}, I_{t_2}, \ldots, I_{t_\ell}$, that contain a point of E_m in their interior ($I_j = [x_{j-1}, x_j] \Rightarrow \text{int } I_j = (x_{j-1}, x_j)$).

So $E_m \subseteq \bigcup_{i=1}^{\ell} I_{t_i} \cup P$.

There exists $x \in E_m \cap \text{int } I_{t_j}$. Then $\omega_f(\text{Int } I_{t_j}) \geq \omega_f(x) \geq 1/m$,

and $\sum_{j=1}^{\ell} \text{Length}(I_{t_j}) = m \sum_{j=1}^{\ell} \frac{1}{m} \text{Length}(I_{t_j}) \leq m \left(\sum_{j=1}^{\ell} \omega_f(\text{int } I_{t_j}) \text{ Length}(I_{t_j}) \right)$

$\underset{\text{apply (*)}}{\leq} m \frac{\varepsilon}{m} = \varepsilon$.

Cover P with intervals of ~~long~~ total length $< \varepsilon$.

We have covered E_m with intervals of total length $< 2\varepsilon$.

$\varepsilon > 0$ is arbitrary so E_m has measure 0.

We have just shown: f Riemann integrable on $[a,b]$ implies f is continuous a.e.

We now seek the partial converse: $f : [a,b] \to \mathbb{R}$ is bounded and continuous a.e. implies f is Riemann integrable on $[a,b]$.

Lemma. ~~Let $O \subseteq \mathbb{R}$, O an interval. Let $f : O \to \mathbb{R}$.~~ Let $f : [a,b] \to \mathbb{R}$. If $\omega_f(x) < \varepsilon$ for $x \in$ ~~O~~ $[a,b]$, then there exists a partition P of O such that $\omega_f(I) < \varepsilon$ on each subinterval I of P

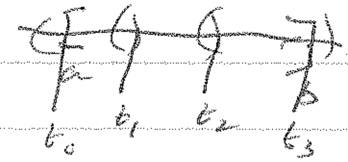
(Recall $P = \{a = x_0, x_1, \dots, x_k = b\}$ so $I = [x_{j-1}, x_j]$ for some $j \in \{1, \dots, k\}$.)

Proof. For each $x \in [a,b]$, find an open interval $I_x \subseteq [a,b]$ such that $x \in I_x$ (a, b are treated with the relative topology) such that $\omega_f(I_x) \le \omega_f(\bar{I}_x) < \varepsilon$. By the Heine-Borel Theorem, there exist finitely many $I_{x_1}, \dots, I_{x_s}$ whose union contains $[a,b]$.

From here you can construct a partition $I_1, \dots, I_k$ of $[a,b]$ and $I_j \subseteq \bar{I}_x$ for some x and so $\omega_f(I_j) \le \omega_f(\bar{I}_x) < \varepsilon$.

Proof by picture to see the construction of the partition

$I_{x_1} \cup I_{x_2} \cup I_{x_3} \supseteq [a,b]$



$P = \{a = t_0, t_1, t_2, t_3 = b\}$.

THM. If $f : [a,b] \to \mathbb{R}$ is bounded and continuous a.e. then f is Riemann integrable on $[a,b]$.

Proof. Let $\varepsilon > 0$. The proof is complete if we construct a partition P of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$ or

$$\sum_{\substack{\text{subintervals } I \\ \text{of } P}} w_f(I)\, \text{length}(I) < \varepsilon .$$

Fix a positive integer m (to be determined later). Let

$$E_m = \{ x : w_f(x) \geq 1/m \}.$$

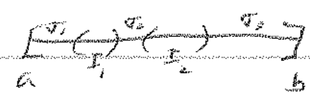
E_m has measure 0.

Find open subintervals which cover E_m with total length $\leq \varepsilon / 4\|f\|$.

Since E_m is closed, (be able to show this) and bounded, E_m is compact. Choose finitely many open intervals $I_1, \dots, I_n$ which cover E_m with total length $\leq \varepsilon/4\|f\|$.

E_m contains no intervals since E_m has measure 0. Thus we can assume without loss of generality that $I_1, \dots, I_n$ do not overlap.

Consider $[a,b] \setminus \bigcup_{i=1}^{n} I_i = \bigcup_{\ell=1}^{K} J_\ell$ where each J_ℓ is a closed interval and $\bigcap_{\ell=1}^{K} J_\ell = \phi$.



Then $w_f(x) < 1/m$ for each $x \in J_\ell$ so partition $J_\ell = \bigcup_{j=1}^{t_j} K_{j_\ell}$ such that $w_f(K_{j_\ell}) < 1/m$.

Construct your partition P as the union of the endpoints of

$$K_{j_\ell},\ j = 1, \dots, t_j,\ \ell = 1, \dots, K, \quad \overline{I_i},\ i = 1, \dots, n.$$

$$U(P,f) - L(P,f) = \sum_{\ell=1}^{k} \sum_{j=1}^{b_j} w_f(K_{j\ell})\, \text{length}(K_{j\ell}) + \sum_{i=1}^{n} w_f(\bar{I}_i)\, \text{length}(\bar{I}_i)$$

$$\le \frac{1}{m} \sum_{\ell} \sum_{j} \text{length}(K_{j\ell}) + 2\|f\| \sum_{i} \text{length}(\bar{I}_i)$$

$$\le \frac{b-a}{m} + \frac{\varepsilon}{2} < \varepsilon, \quad \text{if } m > \frac{b-a}{\varepsilon}. \left(\begin{array}{c} m \text{ is chosen} \\ \text{later.} \end{array}\right)$$

We now move beyond the Riemann integral and begin to develop the Lebesgues integral.

Definition: $s : [a,b] \to \mathbb{R}$ is called a step function if $[a,b] = \bigcup_{i=1}^{k} E_i$ and each E_i is an interval (singleton points are intervals) and s = a constant α_i on each E_i.

Definition: The indicator function. Let $S \subseteq [a,b]$. Define the indicator function of S on $[a,b]$ (the characteristic function of S on $[a,b]$) by

$$I_S(x) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \notin S \text{ or } x \in [a,b] \setminus S. \end{cases}$$

So if S is a step functions $s(x) = \alpha_i$, $x \in E_i$, $[a,b] = \bigcup_{i=1}^{k} E_i$ then

$$s(x) = \sum_{i=1}^{k} \alpha_i I_{E_i} \quad \text{gives a ~~[illegible]~~ representation of } S.$$

Lemma. If $f : [a,b] \to \mathbb{R}$ is Riemann integrable then

$$\int_a^b f = \sup\left\{ \int_a^b s : s : [a,b] \to \mathbb{R} \text{ is a step function satisfying } s(x) \le f(x) \right\}.$$

Proof. Call the sup on the RHS M and note $M \le \int_a^b f$.

Let $\varepsilon > 0$ and choose a partition P of $[a,b]$, $P = \{x_i\}_{i=0}^{n}$ such that

$L(P,f) \geq \int_a^b f - \varepsilon$. Define a step function $s:[a,b] \to \mathbb{R}$ by

$s(x) = \inf_{x \in [x_{i-1}, x_i]} f(x)$ for $x_{i-1} \leq x < x_i$ ($x \in [x_{i-1}, x_i)$),

$i = 1, \ldots, n-1$, and $s(x) = \inf_{[x_{n-1}, x_n]} f(x)$ on $[x_{n-1}, x_n]$.

Then $\int_a^b s = \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f \cdot (x_i - x_{i-1}) = L(P,f) \geq \int_a^b - \varepsilon$.

Thus, $M \geq \int_a^b f - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $M \geq \int_a^b f$. $\square$

Note: if a step function $s = \sum_{i=1}^{K} \alpha_i I_{E_i}$ (E_i an interval)

then $\int_a^b s = \sum_{i=1}^{K} \alpha_i \, \mathrm{Length}(E_i)$.

We seek to extend the concept of length of an Interval to measure of a large class of subset of the reals; in particular we seek a class $\mathcal{M}$ of subsets of the reals and a map $m : \mathcal{M} \to [0, \infty]$ such that if E is an interval, then $E \in \mathcal{M}$ and $m(E) = \mathrm{Length}(E)$.

Then define a "generalized" step function by

$[a,b]$ or $I = \bigcup_{i=1}^{K} E_i$ each $E_i \in \mathcal{M}$, $s = \alpha_i$ on E_i and

$$\int_I s = \sum_{i=1}^{K} \alpha_i \, m(E_i)$$

Then define $\int_I f = \sup\{\int_I s : s \leq f,\ s \text{ a generalized step function}\}$.

Reasonable Properties for the class $\mathcal{M}$

i) $\mathcal{M}$ should contain intervals ; $\phi \in \mathcal{M}$

ii) if $E_1, E_2, \ldots$ (finite or ctble) $\in \mathcal{M}$, then $\bigcup_i E_i \in \mathcal{M}$, $\bigcap_i E_i \in \mathcal{M}$.

iii) $E \in \mathcal{M} \Rightarrow \mathbb{R} \setminus E \in \mathcal{M}$.

iv) $\mathcal{M}$ is translation invariant; that is if $E \in \mathcal{M}$ then

$$E+r = \{ x+r : x \in E \} \in \mathcal{M}.$$

Reasonable Properties for the function (measure) m :

i) $m(E) = \text{length}(E)$, E an interval ; $m(\phi) = 0$.

ii) If $E_1, \ldots$ (finite or ctble) $\in \mathcal{M}$, $E_i \cap E_j = \phi$ if $i \neq j$ (pairwise disjoint)

then $m(\bigcup_i E_i) = \sum_i m(E_i)$.

Defn A collection $\mathcal{F}$ of subsets of $\mathbb{R}$ is called a <u>σ-algebra</u> (a σ-field) if

i) $\phi \in \mathcal{F}$

ii) $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$

iii) $E_1, E_2, \ldots$ (countable) $\Rightarrow \bigcup_i E_i \in \mathcal{F}$.

Note: σ-algebras are closed to finite unions,

write $E_1, \ldots, E_k$ as $E_1, \ldots, E_k, \phi, \ldots, \phi, \ldots$ and apply iii)

σ-algebras are closed to finite intersections, countable intersections

$\bigcap_i E_i$: $\bigcap_i E_i$ in $M \Longleftrightarrow (\bigcap_i E_i)^c$ in M ii

$\bigcup_i E_i^c$ in M iii.

Defn Let $\mathcal{F}$ denote a sigma-algebra. A set function $m : \mathcal{F} \to [0, \infty]$

is called a <u>measure</u> if :

i) $m(\phi) = 0$,

ii) If $E_1, E_2, \ldots$ are pairwise disjoint sets in $\mathcal{F}$, then

$m(\bigcup_i A_i) = \sum_i m(A_i)$ (countable additivity)

We want a σ-algebra $\mathcal{F}$ containing the intervals and a measure m on $\mathcal{F}$ such that $m(E) = \text{length}(E)$ if E is an interval.

Introduce the <u>power set</u> of the reals $\mathcal{P}(\mathbb{R}) = \{\text{set of all subsets of } \mathbb{R}\}$
$= \text{Power set of } \mathbb{R}$.

<u>Defn</u> Lebesgue outer measure is the set function

$$\bar{m} : \mathcal{P}(\mathbb{R}) \to [0, \infty] \text{ such that}$$

$$\bar{m}(E) = \inf\left\{ \sum_{i=1}^{\infty} \text{Length}(E_i) : \bigcup_i E_i \supseteq E, \text{ each } E_i \text{ an open interval} \right\}.$$

Properties of $\bar{m}$

i) If $A \subseteq B$ then $\bar{m}(A) \leq \bar{m}(B)$

ii) If $A \subseteq \bigcup_i A_i$ then $\bar{m}(A) \leq \sum_i \bar{m}(A_i)$ (countable subadditivity)

iii) If E an interval, then $\bar{m}(E) = $ length of E

iv) A has measure $0 \iff \bar{m}(A) = 0$

v) If $A \subseteq \mathbb{R}$ then $\bar{m}(A) = \bar{m}(A + r)$

vi) $\bar{m}(\phi) = 0$.

Proof of ii) Cover A_1 by $E_{11}, E_{21}, \cdots$ (intervals) of total length $< \bar{m}(E_1) + \varepsilon/2$

Cover A_2 by $E_{12}, E_{22}, \cdots$ of total length $< \bar{m}(E_2) + \varepsilon/2^2$

$\vdots$

Cover A_k by $E_{1k}, E_{2k}, \cdots$ of total length $< \bar{m}(E_k) + \varepsilon/2^k$

So $\{E_{ik},\ i, k = 1, 2, \cdots\}$ covers $\bigcup_{i=1}^{\infty} A_i$ with total length $< \sum_i \bar{m}(E_i) + \varepsilon$

and $\bar{m}(\bigcup_i A_i) \leq \sum_i \bar{m}(A_i)$.

Proof of (iii) in the case $E = [a,b]$. $(a-\varepsilon, b+\varepsilon)$ covers E, $\varepsilon > 0 \Rightarrow$

$\bar{m}\,[a,b] \le b-a$.

To show $b-a \le \bar{m}\,[a,b]$ Let $\{E_i\}_{i=1}^{\infty}$ be an open cover of $[a,b]$ consisting of open intervals. Apply the Heine-Borel Theorem and find a finite subcover $E_{i_1}, \ldots, E_{i_k}$.

Employ the Indicator function

$$I_{E_{i_j}}(x) = \begin{cases} 1 & \text{if } x \in E_{i_j} \\ 0 & \text{if } x \notin E_{i_j} \end{cases}, \quad I_{[a,b]} = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{if } x \notin [a,b]. \end{cases}$$

Then $I_{[a,b]} \le I_{E_{i_1}} + \cdots + I_{E_{i_k}}$. Assume $[c,d] \supseteq \bigcup_{j=1}^{k} E_{i_j}$.

Then $\int_c^d I_{[a,b]} \le \sum_{j=1}^{k} \int_c^d I_{E_{i_j}} \Rightarrow b-a \le \sum_{j=1}^{k} (\text{length of } E_{i_j})$

$\le \sum_{j=1}^{\infty} (\text{length of } E_j)$. Thus, $b-a \le \bar{m}\,[a,b]$.

Proof of v will follow by the observation: $\{E_i\}$ covers $A \Leftrightarrow \{E_i + r\}$ covers $A + r$.

For iv) Recall the definition of measure 0, p 179 Goldberg.

A has measure 0 if for each $\varepsilon > 0$, there exist $\{E_i\}_{i=1}^{\infty}$ a countable open cover of A of open intervals and $\sum_i \text{length } E_i < \varepsilon$.

$\bar{m}(A) = 0 \Rightarrow \inf\limits_{\substack{\text{ctble} \\ \text{open covers of } A}} \{ \sum \text{length } E_i \} \Rightarrow$ For $\varepsilon > 0$, find one with $\sum_i \text{length } E_i < \varepsilon$

$\Rightarrow m(A) = 0$. $m(A) = 0 \Rightarrow$ for $\varepsilon > 0$, there is such an open cover with total length $< \varepsilon$

$\Rightarrow \inf \{\text{total length}\} = 0 = \bar{m}(A)$.

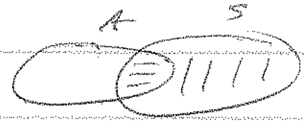
Defn. $A \subseteq \mathbb{R}$ is called a Lebesgue *measurable set* if for every $S \subseteq \mathbb{R}$, $\bar{m}(A \cap S) + \bar{m}(S \setminus A) = \bar{m}(S)$.

That is, A partitions any set S such that the sums of the outer measure of two parts = the outer measure of S.

Theorem. Let $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R})$ denote the collection of all Lebesgue measurable sets. Then $\mathcal{M}$ is a σ-algebra, $\mathcal{M}$ contains all intervals and $\bar{m}$ restricted to $\mathcal{M}$ is a measure m.

(That is, define $m : \mathcal{M} \to [0, \infty]$ by $m(E) = \bar{m}(E)$, $E \in \mathcal{M}$. m is called Lebesgue measure.)

Proof. First we note that if $A \in \mathcal{M}$ then $A^c \in \mathcal{M}$. This will follow from the observation that $S \setminus A = A^c \cap S$



and $S \setminus A^c = S \cap A$.

$$A \in \mathcal{M} \Rightarrow \bar{m}(A \cap S) + \bar{m}(S \setminus A) = \bar{m}(S)$$
$$\Rightarrow \bar{m}(S \setminus A^c) + \bar{m}(A^c \cap S) = \bar{m}(S).$$

To prove the theorem, it is first shown that

1. $\mathcal{M}$ is a translation invariant σ-algebra and $\bar{m}$ restricted to $\mathcal{M}$ is a measure, m, and

2. $\mathcal{M}$ contains all intervals.

Proof of 1. Let $A_1, A_2, \ldots \in \mathcal{M}$. Show $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Let $S \subseteq \mathbb{R}$.

Note $\bar{m}\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \cap S\right) + \bar{m}\left(S \setminus \bigcup_{n=1}^{\infty} A_n\right) \le \bar{m}(S)$ by the subadditivity of $\bar{m}$. (subadditivity over the union of 2 sets $\left(\bigcup_n A_n\right) \cap S$, $S \setminus \bigcup_n A_n$)

First show (*) $\bar{m}(S) \geq \bar{m}\left(S \setminus \bigcup_{k=1}^{n} A_k\right) + \sum_{i=1}^{n} \bar{m}\left(S \cap \left(A_i \setminus \bigcup_{k=1}^{i-1} A_k\right)\right)$ for each n.

By induction $n=1$, $A_1 \in \mathcal{M}$ so $\bar{m}(S) = \bar{m}(S\setminus A_1) + \bar{m}(S \cap A_1)$ and

$$\bar{m}(S) \geq \bar{m}(S\setminus A_1) + \bar{m}(S\cap A_1).$$

Assume (*) is true for $n \geq 1$ an integer. Consider the case $n+1$.

$A_{n+1} \in \mathcal{M}$ so,

$$\bar{m}\left(S \setminus \bigcup_{k=1}^{n} A_k\right) = \bar{m}\left(\left(S\setminus \bigcup_{k=1}^{n} A_k\right) \setminus A_{n+1}\right) + \bar{m}\left(\left(S\setminus \bigcup_{k=1}^{n} A_k\right) \cap A_{n+1}\right)$$

We will do the calculation only for $n=1$. $A_2 \in \mathcal{M}$ so

$$\bar{m}(S\setminus A_1) = \bar{m}\left((S\setminus A_1)\setminus A_2\right) + \bar{m}\left((S\setminus A_1) \cap A_2\right)$$

Recall set algebra calculations $A^c \cap B = B \setminus A$

$(A \cup B)^c = A^c \cap B^c$;

$(A\cap B)^c = A^c \cup B^c$

$$(S\setminus A_1)\setminus A_2 = (S\cap A_1^c)\setminus A_2 = S \cap (A_1^c \cap A_2^c) = S\setminus(A_1 \cup A_2).$$

$$(S\setminus A_1)\cap A_2 = S\cap(A_1^c \cap A_2) = S \cap (A_2 \setminus A_1)$$

$$S\cap(A_1 \cup A_2) = (S\cap A_1) + (S\cap A_2)$$

$$\bar{m}(S) = \bar{m}(S\setminus A_1) + \bar{m}(S\cap A_1)$$

$$= \bar{m}(S\setminus(A_1\cup A_2)) + \bar{m}(S\cap(A_2\setminus A_1)) + \bar{m}(S\cap A_1)$$

$$\geq \bar{m}(S\setminus A_1 \cup A_2) + \bar{m}(S\cap(A_1\cup A_2))$$

since $A_2 \cup A_1 = (A_2 \cup A_1^c) \cup A_1$ and $S\cap(A_1\cup A_2) = S\cap\left((A_2\cap A_1^c)\cup A_1\right)$

$= (S \cap (A_2 \cup A_1^c)) \cup ~~S\cap~~(S \cap A_1)$

With a full induction argument,

$$\bar{m}(S) \geq \bar{m}(S \cap (\bigcup_{i=1}^{n} A_i)) + \bar{m}(S \setminus (\bigcup_{i=1}^{n} A_i)).$$

with subadditivity conclude

$$\bar{m}(S) = \bar{m}(S \cap (\bigcup_{i=1}^{n} A_i)) + \bar{m}(S \setminus (\bigcup_{i=1}^{n} A_i)).$$

We can let $n \to +\infty$ by noting that

$$S \setminus \bigcup_{i=1}^{n} A_i \supseteq S \setminus \bigcup_{n=1}^{\infty} A_n.$$

We already showed $\mathcal{M}$ is translation invariant when we addressed V) in the properties of $\bar{m}$.

To show $\bar{m}$ a measure on $\mathcal{M}$, $\bar{m}(\phi) = 0$. (Any interval of length $< \varepsilon$ covers ϕ.)

To conclude if E_i are pairwise disjoint subsets in $\mathcal{M}$ then

$$\bar{m}(\bigcup_i E_i) = \sum_i \bar{m}(E_i);$$

$\bar{m}(\bigcup_i E_i) \leq \sum_i \bar{m}(E_i)$ for free, subadditivity.

To obtain $\sum_i \bar{m}(E_i) \leq \bar{m}(\bigcup_i E_i)$, one argues that if

$E_1 \cap E_2 = \phi$, find a cover of E_1 with total length L_1
find a cover of E_2 with total length L_2

and then the union of the covers has total length $L_1 + L_2$.

We will say we have completed the proof of 1.

For the proof of 2., we show $\mathcal{M}$ contains all intervals.

Let E be an interval. Let $O \subseteq \mathbb{R}$ be an open interval.

Then $\bar{m}(O) = \bar{m}(O \cap E) + \bar{m}(O \setminus E)$ (this follows from #3 in properties of $\bar{m}$.)

Let $S \subseteq \mathbb{R}$. We are done if we show $\bar{m}(S) = \bar{m}(S \cap E) + \bar{m}(S \setminus E)$ and by subadditivity we are done if we show

$$\bar{m}(S) \geq \bar{m}(S \cap E) + \bar{m}(S \setminus E).$$

Let $\{O_i\}_{i=1}^{\infty}$ denote a family of open intervals covering A.

Then $\{O_i \cap E\}_{i=1}^{\infty}$ covers $S \cap E$ and

$\{O_i \setminus E\}_{i=1}^{\infty}$ covers $S \setminus E$).

Thus

$$\bar{m}(S \cap E) \leq \sum_i \bar{m}(O_i \cap E), \quad \bar{m}(S \setminus E) \leq \sum_i \bar{m}(O_i \setminus E).$$

$$\bar{m}(S \cap E) + \bar{m}(S \setminus E) \leq \sum_i \left(\bar{m}(O_i \cap E) + \bar{m}(O_i \setminus E)\right) = \sum_i \bar{m}(O_i)$$

Take an infimum over the covers $\{O_i\}$ of S and

$$\bar{m}(S \cap E) + \bar{m}(S \setminus E) \leq \bar{m}(S).$$

We are done with 2.

We have stated and proved the Thm

The collection $\mathfrak{M}$ of Lebesgue measurable sets is a σ-algebra and $\bar{m}$, restricted to $\mathfrak{M}$, is a measure.

Defn The class of <u>Borel sets</u> $\mathcal{B}$ is the smallest σ-algebra of subsets of $\mathbb{R}$ containing all intervals.

Lemma. There exists a smallest σ-algebra of subsets of $\mathbb{R}$ containing all intervals.

<u>Proof</u>. Let $\{\mathcal{B}_\alpha\}$ be the family of all σ-algebras of subsets of $\mathbb{R}$ containing all intervals. (There is at least one, $\mathbb{P}(\mathbb{R})$. It is the case that $\mathcal{M} \subseteq \mathbb{P}(\mathbb{R})$ and that the set containment is ~~strict~~ proper, but we have not proved that statement).

Then show $\bigcap_\alpha \mathcal{B}_\alpha$ is a σ-algebra containing all intervals.

1) $\phi \in$ each $\mathcal{B}_\alpha \Rightarrow \phi \in \bigcap_\alpha \mathcal{B}_\alpha$

2) Let $E \in \bigcap_\alpha \mathcal{B}_\alpha$. Then $E \in$ each $\mathcal{B}_\alpha \Rightarrow E^c \in$ each $\mathcal{B}_\alpha \Rightarrow E^c \in \bigcap_\alpha \mathcal{B}_\alpha$.

3) Let $E_1, E_2, \ldots \in \bigcap_\alpha \mathcal{B}_\alpha$. Then $\bigcup_{i=1}^\infty E_i \in$ each $\mathcal{B}_\alpha \Rightarrow \bigcup_{i=1}^\infty E_i \in \bigcap_\alpha \mathcal{B}_\alpha$.

Corollary. $\mathcal{B} \subseteq \mathcal{M}$.

The set containment is proper but we don't show that here.

Remark. Open sets are Borel sets. This follows from a Theorem stated at the bottom of page 136 (Goldberg) and proved at the top of page 137, Goldberg.

Theorem.

Every open subset $\mathcal{O} \subseteq \mathbb{R}$ is a countable disjoint union of open intervals.

Since the Borel sets are closed with respect to countable unions, open sets are Borel sets.

Since Borel sets are closed with respect to complements, closed sets are Borel sets.

Since compact sets in $\mathbb{R}$ are closed (and bdd), compact sets are Borel sets.

A $\mathcal{G}_\delta$ set is a set of the form $\bigcap_{i=1}^{\infty} O_i$, O_i open.
$\mathcal{G}_\delta$ sets are Borel sets.

An $\mathcal{F}_\sigma$ set is a set of the form $\bigcup_i F_i$, F_i closed.
$\mathcal{F}_\sigma$ sets are Borel sets.

Some Notation

$E_i \subseteq \mathbb{R}$, $i = 1, 2, \cdots$, $E \subseteq \mathbb{R}$.

We write $E_n \uparrow E$ if $E_1 \subseteq E_2 \subseteq \cdots$ and $\bigcup_{n=1}^{\infty} E_n = E$.

We write $E_n \Downarrow E$ if $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n=1}^{\infty} E_n = E$.

Properties of Lebesgue measure:

1. m is a translation invariant measure on $\mathcal{M}$
2. m assigns each interval its length

3. If $A, B \in \mathfrak{M}$, $A \subseteq B$, then $m(A) \le m(B)$.
4. If $E_n \uparrow E$ then $m(E_n) \uparrow m(E)$.
5. If $E_n \downarrow E$ and some E_n has finite measure, then $m(E_n) \downarrow m(E)$.
6. $m\left(\bigcup_{n=1}^{\infty} E_n \right) \le \sum_{n=1}^{\infty} m(E_n)$

Proofs of 3, 4, 5, and 6.

3/ Follows immediately from property 1 of outer measure $\bar{m}$

$$m(A) = \bar{m}(A) \le \bar{m}(B) = m(B).$$

or an argument that gives property 3. on any measure

$$B = A \cup (B \setminus A) \quad \text{and} \quad A \cap (B \setminus A) = \phi.$$

So using the property of countable additivity

$$m(B) = m(A) + m(B \setminus A) \ge m(A).$$

4/ $E = E_1 \cup (E_2 \setminus E_1) \cup \cdots \cup (E_n \setminus E_{n-1}) \cup \cdots$

So, again countable additivity

$$m(E) = m(E_1) + m(E_2 \setminus E_1) + \cdots$$
$$= \lim_{n \to \infty} \left(m(E_1) + \cdots + m(E_n \setminus E_{n-1}) \right) = \lim_{n \to \infty} (E_n)$$

5/ If $E_n \downarrow E$ then $E_n^c \uparrow E^c$ and $m(E_n^c) \uparrow m(E^c)$.

$\overset{?}{\Rightarrow} m(E_n) \downarrow m(E)$? If some E_n has finite measure, show this is true.

A counterexample in the case $m(E_n) = +\infty$ for all E_n.

$$E_n = (n, \infty), \quad E_n \downarrow \phi, \quad m(E_n) = +\infty \not\downarrow 0 = m(\phi).$$

b/ On $m\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} m(E_n)$.

Define pairwise disjoints sets E_n' as follows

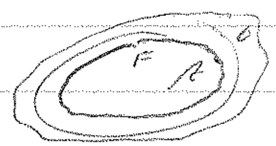
$$E_1' = E_1,\ E_2' = E_2 \setminus E_1,\ E_3' = E_3 \setminus E_1 \cup E_2, \dots, E_n' = E_n \setminus \left(\bigcup_{j=1}^{n-1} (E_j)\right)$$

$\{E_n'\}$ a sequence of pairwise disjoint sets,

$\bigcup_{n=1}^{\infty} E_n' = \bigcup_{n=1}^{\infty} E_n$, $E_n' \subseteq E_n$,

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n') \leq \sum_{n=1}^{\infty} m(E_n).$$

Theorem. Let $A \in \mathcal{M}$. For each $\varepsilon > 0$, there exists $O \subseteq \mathbb{R}$, O open such that $A \subset O$ and $m(O \setminus A) < \varepsilon$ and there exists $F \subseteq \mathbb{R}$, F closed such that $m(A \setminus F) < \varepsilon$.



<u>Proof.</u> Assume $m(A) < \infty$. From the definition of outer measure $\bar{m}$, find open intervals $\{E_j\}$ such that $\bigcup_j E_j \supset A$ and

$$\sum_j m(E_j) < m(A) + \varepsilon.$$

Define $O = \bigcup_j E_j$. $\quad O = (O \setminus A) \cup A$ and $\phi = (O \setminus A) \cap A$

So, $m(O \setminus A) + m(A) = m(O)$ or $m(O \setminus A) = m(O) - m(A)$

$$\leq \sum_j m(I_j) - m(A) < \varepsilon.$$

If $m(A) = +\infty$, write $A = \bigcup_{i=1}^{\infty} A_i$ where A_i has finite measure (set $A_i = A \cap [-i, i]$).

Cover A_i by open O_i such that $m(O_i \setminus A_i) < \varepsilon/2^i$.

Let $O = \bigcup_i O_i$. O covers A and

$$O \setminus A \subseteq \bigcup_i (O_i \setminus A) \quad \text{since}$$

$$O \setminus A = A^c \cap O = A^c \cap (\bigcup_i O_i) \subseteq \bigcup_i (A^c \cap O_i) = \bigcup_i (O_i \setminus A).$$

Thus, $m(O \setminus A) \leq \sum_i m(O_i \setminus A) < \varepsilon$.

To construct the closed set F:

Find O open, $O \supseteq A^c$ and $m(O \setminus A^c) < \varepsilon$.

Set $F = O^c$. F is closed and $F \subseteq A$.

$A \setminus F = A \setminus O^c = A \cap O = O \setminus A^c$

$\Rightarrow m(A \setminus F) = m(O \setminus A^c) < \varepsilon$.

Corollary. Let $A \in \mathfrak{M}$. Then

$$m(A) = \sup\{m(F) : F \text{ closed}, F \subseteq A\}$$
$$= \inf\{m(O) : O \text{ open}, O \supseteq A\}.$$

This Corollary connects these notes with Goldberg's approach of inner and outer measure, section 11.2.

Corollary. Let $A \in \mathfrak{M}$. Then there exists a $\mathcal{G}_\delta$ set G such that $G \supseteq A$ and $m(G \setminus A) = 0$. There exists an $\mathcal{F}_\sigma$ set F such that $F \subseteq A$ and $m(A \setminus F) = 0$.

Proof. Let $\varepsilon_n \downarrow 0$. Find O_n, O_n open $O_n \supseteq A$ and $m(O_n \setminus A) < \varepsilon_n$. Construct O_n' by

$O_1' = O_1$, $O_n' = \bigcap_{i=1}^{n} O_i$. Then $O_n' \downarrow G = \bigcap_{n=1}^{\infty} O_n'$

and $(O_n' \setminus A) \downarrow (G \setminus A)$ and $m(G \setminus A)$ is finite.

So, $0 \le m(G \setminus A) = \lim_{n\to\infty} m(O_n' \setminus A) \le \lim_{n\to\infty} \varepsilon_n = 0$.

For you, construct an F_σ set F such that $F \subset A$ and $m(A \setminus F) = 0$.

Theorem. The uniqueness of Lebesgue measure.
If m^* is a measure on $\mathcal{M}$ which assigns each interval its length, then $m^* = m$ on $\mathcal{M}$.

Proof. Let m' denote a measure on $\mathcal{M}$ assigning each interval its length. Show $m = m'$. Note m and m' agree on open sets since open sets are countable disjoint unions of open intervals.

Let K be compact. Let $O \supseteq K$, O open. Then

$$K = \underbrace{O}_{\text{open}} \setminus \underbrace{(O \setminus K)}_{\text{open}} \Rightarrow m(K) = m(O) - m(O \setminus K)$$
$$= m'(O) - m'(O \setminus K) = m'(K).$$

So, m, m' agree on compact sets.

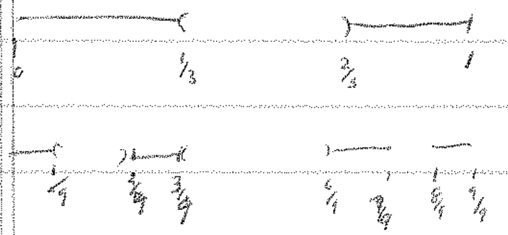
Let $A \in \mathcal{M}$, A bdd and find $F \subseteq A$, F closed, $m(A) < m(F) + \frac{\varepsilon}{2}$,
$A \subseteq O$, O open, $m(O) < m(A) + \frac{\varepsilon}{2}$.

Then $m(O) - m(F) = m(O) - m(A) + m(A) - m(F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$.

$$F \subseteq A \subseteq O \Rightarrow \overset{m(F)''}{m'(F)} \le m'(A) \le m'(O) = m(O)$$
$$\Rightarrow m(F) \le m'(A) \le m(O) \text{ and}$$
$$m^*(F) \le m^*(A) \le m(O)$$
$$\Rightarrow |m(A) - m'(A)| < \varepsilon. \quad \varepsilon > 0 \text{ is arbitrarily selected} \Rightarrow m'(A) = m(A).$$

To finish the arguement, assume $A \in \mathcal{M}$, A not bounded.
Write $A = \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{M}$, A_i bdd, $A_i \cap A_j = \emptyset$ if $i \neq j$.

A construction of the Cantor set. The Cantor set $C \subseteq \mathbb{R}$ is an example of an uncountable set with measure 0.



First stage $\quad I_0 = [0, \tfrac{1}{3}]$, $\quad I_1 = [\tfrac{2}{3}, 1]$

2nd stage $\quad I_{00} = [0, \tfrac{1}{9}]$, $I_{01} = [\tfrac{2}{9}, \tfrac{3}{9}]$ $\quad I_{10} = [\tfrac{6}{9}, \tfrac{7}{9}]$, $I_{11} = [\tfrac{8}{9}, \tfrac{9}{9}]$

Proceed inductively and at the nth stage, construct 2^n sets

$$I_{j_1 \cdots j_n}, \quad \text{where } j_i = 0 \text{ or } 1,\ i = 1, \cdots, n.$$

$I_{j_1 \cdots j_n}$ is a closed interval with length $1/3^n$,

$$I_{j_1 \cdots j_{n-1} j_n} \subseteq I_{j_1 \cdots j_{n-1}}, \quad I_{j_1 \cdots j_n} \cap I_{j_1' \cdots j_n'} \text{ if } (j_1 \cdots j_n) \neq (j_1' \cdots j_n').$$

Let $C_n = \bigcup_{(j_1 \cdots j_n)} I_{j_1 \cdots j_n}$ (the union of all 2^n intervals of length $= 1/3^n$).

Thus, $C_{n+1} \subseteq C_n$. Let $C = \bigcap_{n=1}^{\infty} C_n$. C is called the Cantor set.

We show
i) C is closed
ii) C has measure 0
iii) C is uncountable.

i) C is closed since arbitrary intersections of closed sets is closed.

ii) C has measure 0 since $m(C_n) = (\tfrac{2}{3})^n \to 0$ as $n \to \infty$ and $\bigcap_{n=1}^{\infty} C_n = C$.

iii) C is uncountable.

Proof is to exhibit a 1-1, onto relationship between C and an uncountable set.

Define $S = \{ (j_1, j_2, j_3, \ldots, j_n, \ldots), \text{ each } j_i = 0 \text{ or } 1 \}$.

Map $(j_1, j_2 \ldots) \to x$ where $x \in \bigcap_{n=1}^{\infty} I_{j_1 \ldots j_n}$.

Note that by the Heine-Borel Theorem, there exists x such that $\{x\} = \bigcap_{n=1}^{\infty} I_{j_1 \ldots j_n}$.

Clearly S is 1-1 and onto

The same construction to show the reals are uncountable can be used to show S is uncountable.

Assume you can count S and so count them

$s_1 = (j_{11}\ j_{12} \ldots$

$s_2 = (j_{21}\ j_{22} \ldots$

$s_3 = (j_{31}\ j_{32} \ldots$

$\vdots$

Construct $s \in S$ as follows

$s = (j_1\ j_2 \ldots)$

For each ~~k~~ n, if ~~$s_{nn} = 1$~~ $j_{nn} = 1$ then ~~x_n~~ $j_n = 0$ and if ~~s_{nn}~~ $j_{nn} = 0$ then ~~x_n~~ $j_n = 1$.

Then $s \in S$ and $s \neq s_k$ for all k because $j_k \neq j_{kk}$.

Conclude: we did not exhaust S by counting them so S is unctble.

There exist subsets of $\mathbb{R}$ that are not Lebesgue measurable and we construct one here.

Define an equivalence relation $\sim$ on $[0,1]$ as follows:

$$x \sim y \iff x-y \text{ is rational.}$$

Partition $[0,1]$ into equivalence classes $\{B_\alpha\}$ meaning

$x, y \in B_\alpha \iff x \sim y$.

use of axiom of choice

For each B_α, select $P_\alpha \in B_\alpha$ and define $P = \bigcup_\alpha \{P_\alpha\}$.

We show $P \notin \mathcal{M}$ by contradiction, so assume $P \in \mathcal{M}$.

Let $\{r_1, r_2, \ldots\}$ denote the rationals in $[-1,1]$.

Define sets $E_1, E_2, \ldots$ by $E_i = P + r_i$ (meaning $x \in E_i$ if $x \in p + r_i$ for some $p \in P$).

Note $E_i \cap E_j = \phi$ if $i \neq j$. To see this, assume $x \in E_i \cap E_j$ for some $i \neq j$.

Then $x = p + r_i$ for some $p \in P$

$= p' + r_j$ for some $p' \in P$

and $p' - p = r_j - r_i$ (rational) and so $p' \sim p$ which is a contradiction.

So, $E_i \cap E_j = \phi$ if $i \neq j$.

Note $\bigcup_i E_i \subseteq [-1,2]$ since each $0 \leq p \leq 1$ and $-1 \leq r_i \leq 1$.

So $[0,1] \subseteq \bigcup_i E_i \subseteq [-1,2]$.

If $P \in \mathcal{M}$, then each $E_i \in \mathcal{M}$ and $m(E_i) = m(P) \stackrel{\text{some}}{=} \alpha$.

If $\alpha = 0$, this contradicts $[0,1] \subseteq \bigcup_i E_i$ since $1 \leq \sum_i 0 = 0$ is a contradiction.

If $\alpha > 0$, this contradicts $\bigcup_i E_i \subseteq [-1,2]$ since $\infty = \sum_i \alpha \leq 3$ is a contradiction.