Homework 7
due Nov 10, 10pm

**1.** Let $R = \begin{pmatrix} 0 & 1 & -1 & 0 & 17 \\ 0 & 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, and consider the equation $R\mathbf{x} = \mathbf{b}$, with $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix}$.

a) What conditions must the entries of $\mathbf{b}$ satisfy in order for $A\mathbf{x} = \mathbf{b}$ to have a solution?
b) What is the nullspace of $R$? (Write it in terms of a basis.)
c) Assuming that the conditions in (a) are satisfied, what is a particular solution $\mathbf{x}_p$ to $A\mathbf{x} = \mathbf{b}$? (You can set the free variables to anything when finding $\mathbf{x}_p$; but please set them to zero here.)
d) What is the full solution to $A\mathbf{x} = \mathbf{b}$?

There are two good ways to write the full solution. If a basis for $N(A)$ consists of some $\mathbf{v}_3$ and $\mathbf{v}_5$, you can write

$$\mathbf{x} = x_3\mathbf{v}_3 + x_5\mathbf{v}_5 + \mathbf{x}_p\,.$$

Alternatively, you can write

$$\mathbf{x} \in \langle \mathbf{v}_3, \mathbf{v}_5 \rangle + \mathbf{x}_p\,.$$

In the second option, you are thinking of $N(A) + \mathbf{x}_p = \langle \mathbf{v}_3, \mathbf{v}_5 \rangle + \mathbf{x}_p$ as a *set*, and saying that $\mathbf{x}$ must belong to this set.

**2.** Let $R = \begin{pmatrix} 1 & 0 & 42 & 0 & -9 \\ 0 & 1 & 300 & 0 & 10 \\ 0 & 0 & 0 & 1 & 11/5 \end{pmatrix}$. Write down the full solution to $R\mathbf{x} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

(Please set free variables to zero when finding an $\mathbf{x}_p$.)

**3.** Let $A = \begin{pmatrix} 1 & 5 & 2 & -4 \\ 1 & 5 & 3 & -1 \\ 1 & 5 & 2 & -3 \\ 0 & 0 & -2 & -6 \end{pmatrix}$ and consider the equation $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$.

a) Write down the augmented matrix $(A\,|\,\mathbf{b})$ and use row operations to transform it to $(R\,|\,\mathbf{b}')$.
b) Which invertible matrix $M$ implements these row operations? In other words, find the explicit $M$ such that $MA = R$ (and $M\mathbf{b} = \mathbf{b}'$).

There are two good ways to do this. Either keep track of all the elimination matrices and multiply them out. Or start with an augmented matrix $(A\,|\,I)$ and use row operations to transform it to $(R\,|\,M)$.

c) What condition must the entires of $\mathbf{b}$ satisfy in order for a solution to exist?
d) Assuming the condition in (c), write down the full solution.

(Please set free variables to zero when finding an $\mathbf{x}_p$.)

**4.** For each of the following matrices, write down the rank and the dimensions of $N(R)$, $C(R)$, $N(R^T)$, and $C(R^T)$:

$$\text{a) } R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{b) } R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{c) } R = \begin{pmatrix} 0 & 1 & -1 & 0 & 17 \\ 0 & 0 & 0 & 1 & 20 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{d) } R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{e) } R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

**5.** Suppose we are trying to solve $A\mathbf{x} = \mathbf{b}$, and find that $A$ has a reduced row-echelon form $R$. For each of the $R$'s from problem 4, state which of the following are possible: $A\mathbf{x} = \mathbf{b}$ has no solutions; $A\mathbf{x} = \mathbf{b}$ has a single, unique solution; $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. (More than one of these may be possible.)

**6.** Suppose that $A$ is a $p \times n$ matrix and $MA = R$, for an invertible matrix $M$. True or false:

a) The span of the rows of $A$ is a subspace of $\mathbb{R}^p$.
b) The span of the rows of $A$ is identical to the span of the rows of $R$.
c) The span of the rows of $A$ has the same dimension as the span of the rows of $R$.
d) The span of the columns of $A$ is identical to the span of the columns of $R$.
e) The span of the columns of $A$ has the same dimension as the span of the columns of $R$.
f) If $A$ is invertible then $R$ is invertible.

Note that the answers to these questions do *not* require $R$ to be the rref of $A$. They hold for any pair of matrices $A, R$ related by $MA = R$.

**7.** Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

a) What is the rank of $A$?

Let $B$ be a $4 \times 6$ matrix written in terms of $2 \times 2$ blocks as $B = \begin{pmatrix} A & A & A \\ 3A & 3A & 3A \end{pmatrix}$.

b) What is the rank of $B$?

**8.** Suppose that $A$ is a $3 \times 5$ matrix and that $B$ is a $4 \times 3$ matrix — so that the product $BA$ is a $4 \times 5$ matrix.

a) If $\text{rank}(A) = 2$ and $\text{rank}(B) = 3$, what are the minimum and maximum possible ranks of $BA$?
b) If $\text{rank}(A) = 2$ and $\text{rank}(B) = 2$, what are the minimum and maximum possible ranks of $BA$.
c) If you know nothing about the ranks of $A$ and $B$, what are the minimum and maximum possible ranks of $BA$?

**9.** Let $A = \begin{pmatrix} 0 & 2 & 1 & 7 & 1 \\ 0 & -4 & 1 & -5 & -4 \\ 0 & 2 & 7 & 25 & -4 \\ 0 & -2 & -4 & -16 & 0 \end{pmatrix}$.

Not to turn in: find the rref's of $A$ *and* $A^T$, and show that they look like

$$R = \text{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Q = \text{rref}(A^T) = \begin{pmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In the case of $A^T$, you must begin with a permutation because the first pivot is zero; use the permutation that swaps the 1st and 5th rows. (The rest of this problem is to turn in:)

a) What is the rank $r$ of $A$?
b) True or false: The number of pivots in $R$ must equal the number of pivots in $Q$.
c) What are the dimensions of $N(A)$, $C(A)$, $N(A^T)$, and $C(A^T)$?
(This should not require any computations!)

d) How many solutions does $A\mathbf{x} = 0$ have: 0, 1, or $\infty$? How many conditions must we impose on entries of $\mathbf{b}$ in order for $A\mathbf{x} = \mathbf{b}$ to have a solution?
(This should not require any computations!)

e) Find a basis for the nullspace, writing $N(A) = \langle \mathbf{v}_1, \mathbf{v}_2, ... \rangle$ for some vectors $\mathbf{v}_i$.
f) Find a basis for the column space $C(A) = \langle \mathbf{c}_1, \mathbf{c}_2, ... \rangle$.

To do this, use the same reasoning as on the last problem of HW 6. Namely, notice that the column space of $A$ is the same as the *row* space of $A^T$. The span of the rows of $A^T$ is equivalent to the span of the rows of $Q$ (since $A^T$ and $Q$ are related by row operations). Moreover, the nonzero rows of $Q$ are manifestly independent. So: just take the $\mathbf{c}$'s to be nonzero rows of $Q$, transposed back into column vectors!

g) Find a basis for the column space $C(A^T) = \langle \mathbf{d}_1, \mathbf{d}_2, ... \rangle$.

Remember this is equivalent to the row space of $A$. So a good basis can just be read off from the nonzero rows of $R$, transposed back into columns!

h) Find a basis for the nullspace $N(A^T) = \langle \mathbf{w}_1, ... \rangle$.
i) Check that $N(A) \perp C(A^T)$, by showing that every $\mathbf{v}$ is perpendicular to every $\mathbf{d}$. Similarly, check that $N(A^T) \perp C(A)$ by showing that every $\mathbf{w}$ is perpendicular to every $\mathbf{c}$.

j) Suppose we want to solve $A\mathbf{x} = \mathbf{b}$, with $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$. Use what you know about $N(A^T)$ to write down the condition(s) that the entries $\mathbf{b}$ must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to have a solution.

In particular: do *not* do this via an augmented matrix! Use the fact that a vector $\mathbf{b}$ is in $C(A)$ if and only if it's perpendicular to every basis vector of $N(A^T)$.

**10.** Let $A = \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 2 & 6 & -4 \\ 1 & 2 & 1 & 0 \end{pmatrix}$.

a) Find the rref $R$ of $A$ and the rref $Q$ of $A^T$.
b) What's the rank of $A$?
c) What are the dimensions of $N(A)$, $C(A)$, $N(A^T)$, and $C(A^T)$?
d) Find a basis for $N(A)$.
e) Find a basis for $C(A)$. (Please use the nice basis coming from $Q$.)
f) Find a basis for $C(A^T)$. (Please use the nice basis coming from $R$.)
g) Find a basis for $N(A^T)$.
h) Show that $N(A) \perp C(A^T)$ and $N(A^T) \perp C(A)$.
i) Use (g) to write down the conditions that $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to have a solution.

*Final remark:* When thinking of $A$ abstractly as a linear map $\mathbb{R}^n \xrightarrow{A} \mathbb{R}^p$ rather than a matrix, there is some different terminology that's common for describing the four subspaces:

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|---|---|---|---|---|
| (right) nullspace, | $N(A)$ | $=$ | kernel of $A$, | $\ker(A)$ |
| column space, | $C(A)$ | $=$ | image of $A$, | $\operatorname{im}(A)$ |
| left nullspace, | $N(A^T)$ | $=$ | cokernel of $A$, | $\operatorname{coker}(A)$ |
| row space, | $C(A^T)$ | $=$ | coimage of $A$, | $\operatorname{coim}(A)$ |

The transpose $\mathbb{R}^n \xleftarrow{A^T} \mathbb{R}^p$ is often called the "dual map," and properties of the dual map (like the cokernel and coimage) get an extra "co-" as a prefix.