Homework 10; Due Wednesday, 11/9/2016

Medium Justification Questions. Provide brief justifications for your responses.

Question 1. Define the function $L : \mathbb{R} \to \mathbb{R}$ as $L(x) = e^x$. Is L a linear function? Briefly justify.

Solution. Write your solutions here!

Question 2. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear function such that $L((1,1)) = (1,0)$ and $L((-1,1)) = (0,1)$. Using properties of a linear function, find $L((0,2))$.

Question 3. Let V be the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$. Define addition by

$$(f+g)(x) = f(x) + g(x)$$

and scalar multiplication by

$$(cf)(x) = cf(x)$$

Prove that V is a vector space over $\mathbb{R}$.

Question 4. Let

$$V = \{f\colon \mathbb{R} \to \mathbb{R} \mid f(2) = 0\}.$$

Is V a vector space over $\mathbb{R}$ with addition and scalar multiplication defined in question 3? Briefly justify your answer. If you believe the answer is "yes", then you may use that the set of all functions $f\colon \mathbb{R} \to \mathbb{R}$ is a vector space from the previous problem, and use Proposition 8.5 of the additional notes. If you believe the answer is "no", then choose one of the ten vector space requirements that fails for V and briefly explain why it fails.

Question 5. Define the set $V = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 3\}$. Sketch a picture of the set V inside of the plane, $\mathbb{R}^2$. Is V a vector space over the scalar field $\mathbb{R}$? Briefly justify your answer. If you believe the answer is "yes", then you may use Proposition 8.5 of the additional notes. If you believe the answer is "no", then choose one of the ten vector space requirements that fails for V and briefly explain why it fails.

Question 6. For each of the following, answer "yes" or "no", and briefly justify your answer:

(i) Is $D : \mathbb{P}_3 \to \mathbb{P}_2$, with $D(p) = \frac{dp}{dx}$, a linear function?

(ii) Is $E : \mathbb{P}_2 \to \mathbb{R}$, with $E(p) = p(0)$, a linear function?

(iii) Is $f\colon \mathbb{R} \to \mathbb{R}$, with $f(x) = 2x + 1$, a linear function?

Complete Justification Questions. Provide complete justifications for your responses.

Question 7. Assume that V and W are both vector spaces over $\mathbb{R}$ and that $L : V \to W$ is a linear function. Define the set

$$K = \{v \in V \mid L(v) = 0\}.$$

Prove that K is itself another vector space by stating how you know that $K \subseteq V$ and use definition 8.4 and proposition 8.5 (instead of checking all 10 properties in definition 8.1).

Question 8. Assume that V is a vector space over $\mathbb{R}$, with $V \neq \{0\}$, and $L\colon V \to V$ is a linear map such that $L(L(v)) = 0$ for all $v \in V$. Prove that L is not injective.

Question 9. Define the matrix, $A = \begin{pmatrix} 1 & 0 \\ 2 & c \end{pmatrix}$, and define the map $L : \mathbb{R}^2 \to \mathbb{R}^2$ via matrix multiplication. That is, if $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, then L is defined as

$$L(v) = Av = \begin{pmatrix} 1 & 0 \\ 2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Prove that L is surjective if and only if $c \neq 0$.

Question 10. Let $L : \mathbb{P}_2 \to \mathbb{P}_2$ be a **linear** function.
Prove that L is injective if and only if the following two sets are equal:

$$\{p \in \mathbb{P}_2 \ : \ L(p) = 0\} = \{0\}.$$

Note, in both occurrences in the previous equation, 0 is the zero polynomial, that is the function, $z : \mathbb{R} \to \mathbb{R}$, such that $z(t) = 0$ for all $t \in \mathbb{R}$. It is, of course, also the polynomial whose coefficients are all zero.
Here are some steps:

1. recognize that this is of the form $A \iff B$, and so you must provide a separate proof for each of $A \implies B$ and $B \implies A$.
2. it could be helpful to try to do an argument by contrapositive to establish that $A \implies B$ is true.
3. when working on $B \implies A$, suppose you are investigating the possibility that $p, q \in \mathbb{P}_2$ and $L(p) = L(q)$. What does the fact that L is linear say about $L(p) - L(q)$?