Homework 6

due Nov 3, 10pm

Recall: given a vector space (or a subspace) $\mathbb{V}$, a *basis* for $\mathbb{V}$ is a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ that a) span $\mathbb{V}$ and b) are independent.

The mathematical definition of *dimension* $\dim(\mathbb{V})$ is the number of vectors in a basis (any basis). A basic theorem is that every basis has exactly the same number of vectors, so it makes sense to count them and call this number $\dim(\mathbb{V})$. You can use this result/definition backwards: if you're trying to find a basis for a space $\mathbb{V}$ and already happen to know what its dimension, then you know how many vectors $\mathbf{v}_i$ to look for.

If you're given a vector (or a subspace) space $\mathbb{V}$ and asked to find a basis, you can always use the following algorithm. First, choose any nonzero vector $\mathbf{v}_1$ in $\mathbb{V}$. Find its span $\langle \mathbf{v}_1 \rangle$. Ask: does $\langle \mathbf{v}_1 \rangle = \mathbb{V}$? (Does the span of $\mathbf{v}_1$ cover all of $\mathbb{V}$?) If so, you're done. If not, there exist vectors in $\mathbb{V}$ that are not in $\langle \mathbf{v}_1 \rangle$, so choose one; call it $\mathbf{v}_2$. Ask: does $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \mathbb{V}$? If so, you're done. If not, choose a $\mathbf{v}_3$ in $\mathbb{V}$ that's not in the span $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$... and continue.
In this class, the process will usually stop at some point.
It's the point when you've hit the dimension of $\mathbb{V}$.
(If the process does not stop, then $\dim(\mathbb{V}) = \infty$.)

1. Consider the plane $\mathbb{P}$ in $\mathbb{R}^4$ containing all vectors of the form $\begin{pmatrix} a+b \\ a \\ 3b \\ b \end{pmatrix}$, as a and b range over real numbers. Since this plane contains the origin ($\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ comes from setting $a = 0$ and $b = 0$), and we know that any plane containing the origin is a subspace, $\mathbb{P}$ is a subspace of $\mathbb{R}^4$. Find a basis for $\mathbb{P}$.

2. Let $\mathbb{L} \subseteq \mathbb{R}^2$ be the line containing all vectors perpendicular to $\mathbf{t} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

a) Use the definition of a subspace to show that $\mathbb{L}$ is a subspace of $\mathbb{R}^2$.

(Hint: you must show, for arbitrary vectors $\mathbf{v}$ and $\mathbf{w}$ and a scalar c, that if $\mathbf{v}$ is in $\mathbb{L}$ then $c\mathbf{v}$ is in $\mathbb{L}$; and that if $\mathbf{v}$ and $\mathbf{w}$ are in $\mathbb{L}$ then $\mathbf{v} + \mathbf{w}$ are in $\mathbb{L}$.)

More directly (though not for this problem): you can use the knowledge that any line through the origin is a subspace, and simply check that $\mathbb{L}$ contains the origin to convince yourself that $\mathbb{L}$ is a subspace. $\mathbb{L}$ contains the origin because $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \mathbf{t} = 0$.

b) Find a basis for $\mathbb{L}$.

3. Let $\mathbb{M}$ be the vector space of 2×2 matrices, and let $\mathbb{W}$ contain all matrices whose diagonal matrices sum to zero, *i.e.* matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a + d = 0$. The matrices in $\mathbb{W}$ are called "traceless" matrices.

a) Find a basis for $\mathbb{M}$.

b) Use the definition of subspace to show that $\mathbb{W}$ is a subspace of $\mathbb{M}$.

c) What do you expect to be the dimension of $\mathbb{W}$? (This should be the dimension of the $\mathbb{M}$, minus the number of conditions or relations that are obeyed in $\mathbb{W}$.)

d) Find a basis for $\mathbb{W}$.

4. Let $\mathbb{P} \subseteq \mathbb{R}^3$ be the set of vectors perpendicular to $\mathbf{t} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Let $\mathbb{P}_{xy}$ by the xy plane, *i.e.* the set of vectors of the form $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ for any real numbers x, y. Both $\mathbb{P}$ and $\mathbb{P}_{xy}$ are subspaces.

a) Find a basis for $\mathbb{P}$.

b) Find a basis for $\mathbb{P}_{xy}$.

c) Find a basis for the intersection of $\mathbb{P}$ and $\mathbb{P}_{xy}$: the set of vectors that are *both* in $\mathbb{P}$ and $\mathbb{P}_{xy}$.

5. Strang Section 3.4, problem 41, reworded here:

Consider the six 3×3 permutation matrices:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

a) Show that P_1, P_2, P_3, P_4, P_5 are independent (as vectors in the vector space of all 3×3 matrices). Do this by writing down the matrix $c_1P_1 + c_2P_2 + c_3P_3 + c_4P_4 + c_5P_5$ and showing that if this matrix is zero then necessarily $c_1 = c_2 = ... = c_5 = 0$.

The matrices P_1, P_2, P_3, P_4, P_5 form a basis for a vector space $\mathbb{W}$, which contains 3×3 matrices such that the entries each row and each column all sum to the same number.

b) Notice that I is in $\mathbb{W}$. Find a linear combination of the P's that equals I.

Recall that the *nullspace* or *kernel* of a linear map A from $\mathbb{R}^n$ to $\mathbb{R}^p$ is the subspace of $\mathbb{R}^n$ containing all vectors $\mathbf{x}$ such that $A\,\mathbf{x} = 0$. In other words, it is the space of all solutions to $A\,\mathbf{x} = 0$. In the book it is denoted $N(A)$. Often in mathematics one finds the notation $\ker(A)$ for "kernel."

In class and in Strang Section 3.2, you learned how to find the reduced row-echelon (rref) form of a matrix A. The basic idea is to do row operations on A — in other words, multiply A on the left by lower triangular matrices, permutation matrices, upper triangular matrices, and diagonal matrices — to get

$$(DF...FE..E)A = R$$

Here $(DF...FE..E)$ encodes the row operations, and R is the reduced row-echelon form. The matrix R has

- all its pivots set to 1
- all other entries in a pivot column set to 0
- any rows not containing a pivot set entirely to zero
- any remaining entries in a column not containing a pivot undetermined.

One of the benefits of R is that it allows us to easily find the nullspace of A. The key property is that, because the row operations $(DF...FE..E)$ are *invertible*,

$$A\mathbf{x} = 0 \qquad \text{if and only if} \qquad R\mathbf{x} = 0\,.$$

And $R\mathbf{x} = 0$ is relatively easy to solve.

6. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 10 \end{pmatrix}$.

a) Find the reduced row-echelon form R of A.

Hint: in this case, it will look like $R = \begin{pmatrix} 1 & \square & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

b) Use back-substitution to find all solutions $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ to $A\mathbf{x} = 0$. You should find here that x_2 is a free variable (because the second column of R does not contain a pivot). The remaining variables x_1 and x_3, whose columns do contain pivots, are determined in terms of x_2.

c) Find a basis for the nullspace $N(A)$.

7. Repeat the steps in Problem 6 for $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

8. Repeat the steps in Problem 6 for $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$.

9. Repeat for $A = \begin{pmatrix} 2 & -4 \\ 6 & -11 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}$.

10. Repeat for $A = \begin{pmatrix} 2 & -4 \\ 6 & -12 \\ 1 & -2 \\ -1 & 2 \end{pmatrix}$.

The rref also gives a quick way to describe the span of a set of vectors $\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \rangle$. Here's how. First arrange the vectors as rows of A

$$A = \begin{pmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_n^T \end{pmatrix}.$$

Then suppose that the rref of A looks like

$$R = \begin{pmatrix} \mathbf{w}_1^T \\ \mathbf{w}_2^T \\ \vdots \\ \mathbf{w}_k^T \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The claim is that the $\mathbf{w}$'s span exactly the same space as the $\mathbf{v}$'s: $\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \rangle = \langle \mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_k \rangle$. This is because all the row operations we did to get R are invertible. In other words, we have

$$MA = R \qquad \text{and also} \qquad A = M^{-1}R$$

for some invertible matrix M that encodes all the row operations. The second form $A = M^{-1}R$ says exactly how to take linear combinations of the rows of R in order to reconstruct the rows of A. So *every* $\mathbf{v}$ can be written in terms of the $\mathbf{w}$'s. (And vice versa).
Now because the $\mathbf{w}$'s are rows of a reduced row-echelon matrix, they are *also* an independent set of vectors. Therefore, they form a basis for $\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \rangle$.

So: at once the rref R gives us a basis for $\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \rangle$, and tells us that the dimension of the space $\langle \mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \rangle$ must be k.

You have had lots of problems in the past that asked you to find/describe the span of a collection of vectors. Now you have a systematic and efficient way to do it.

11. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ -2 \\ 6 \\ 5 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} -1 \\ 1 \\ -6 \\ -7 \end{pmatrix}$. Write down a matrix A as above whose rows are the $\mathbf{v}^T$'s. Find the rref R of A. What are the nonzero rows $\mathbf{w}_1^T, \mathbf{w}_2^T, ...$ of R? What's the resulting basis for the span $\langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \rangle \subseteq \mathbb{R}^4$?