Infinite Series (Week 7)

Definition. If $\sum|a_n|$ converges, then the series $\sum a_n$ converges is said to **converge absolutely** (or to be **absolutely convergent**). If $\sum a_n$ converges but $\sum|a_n|$ diverges, then $\sum a_n$ is said to **converge conditionally** (or be **conditionally convergent**). **(Def. 2.5.12, Lebl)**

Theorem (Alternating Series Test)
If (a_n) is a decreasing sequence of positive numbers and lim $a_n = 0$,
then the alternating series $\sum(-1)^{n+1}a_n$ converges.

Example: The alternating harmonic series $\sum(-1)^{n+1}\left(\frac{1}{n}\right)$ converges conditionally.

Theorem (Root Test) Given a series $\sum a_n$, let $\alpha = \limsup|a_n|^{1/n}$.
(a) If $\alpha < 1$, then the series converges absolutely.
(b) If $\alpha > 1$, the series diverges.
(c) Otherwise $\alpha = 1$, and the test gives no information about convergence or divergence.

Definition. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. **(Section 2.6.5, Lebl)**

The series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

is called a **power series (centered at 0)**. The number a_n is called the ***n*th coefficient** of the series.

Definition. More generally, a **power series centered at x_0** has the form $\sum a_n(x - x_0)^n$.

Example: $\sum x^n$ is a geometric power series which converges iff $|x| < 1$.

Root Criterion for Power Series. Let $\sum a_n x^n$ be a power series centered at 0 and let $R = \limsup |a_n|^{1/n}$.
Define ρ by

$$\rho = \begin{cases} \frac{1}{R}, & if\ 0 < R < +\infty \\ +\infty, & if\ R = 0 \\ 0, & if\ R = +\infty \end{cases}$$

Then the series converges absolutely whenever $|x| < \rho$ and diverges whenever $|x| > \rho$. (When $\rho = \infty$, we take this to mean that the series converges absolutely for all real x. When $\rho = 0$, then the series converges only at 0.) **(Prop. 2.6.9 and 2.6.10, Lebl)**

ρ is called the **radius of convergence**.

When $\rho = 0$, the power series converges only at 0.
When $0 < \rho < \infty$, the power series converges for x in the interval $(-\rho, \rho)$. (Convergence/divergence at the endpoints $-\rho$ and ρ must be determined separately. The theorem does not tell us anything about the behavior of the series for $x = \rho$ or $x = -\rho$.)
When $\rho = \infty$, the power series converges for all real numbers x, that is, for any x in $(-\infty,\infty)$.

The set of values for which the power series converges is called the **interval of convergence**. Note that this interval has one of the following categorizations: open, or closed, or half-open (including one of the two endpoints), depending upon the power series in question.

Example: For $\sum x^n$, the radius of convergence is 1. The interval of convergence is (-1, 1).

Ratio Criterion for Power Series. Given power series $\sum a_n x^n$ centered at 0,

If $\lim \left|\frac{a_{n+1}}{a_n}\right|$ exists, set $R = \lim \left|\frac{a_{n+1}}{a_n}\right|$. (The limit could be infinite.)

The radius of convergence ρ is

$$\rho = \begin{cases} \frac{1}{R}, & if\ 0 < R < +\infty \\ +\infty, & if\ R = 0 \\ 0, & if\ R = +\infty \end{cases}$$

Example: For $\sum \frac{1}{n} x^n$, the radius of convergence is 1. The interval of convergence is [-1, 1).

Example: For $\sum \frac{1}{n^2} x^n$, the radius of convergence is 1. The interval of convergence is [-1, 1].

Example: For $\sum \frac{1}{n!} x^n$, the radius of convergence is ∞. The interval of convergence is (-∞,∞).

Example: For $\sum n^n x^n$, the radius of convergence is 0. The interval of convergence is {0}.

REMARK: For a power series centered at x_0, the same convergence tests apply, with the interval of convergence centered at x_0.

Example: For $\sum (x-1)^n$, x_0 = 1, the radius of convergence is 1, and the interval of convergence is

$(x_0 - 1, x_0 + 1) = (1 - 1, 1 + 1) = (0, 2)$.