Sequences and Series of Functions

Pointwise and Uniform Convergence

Definition. Let (f_n) be a sequence of functions defined on a subset S of **R**.
Then (f_n) **converges pointwise on *S*** if for each x in S, the sequence of numbers $(f_n(x))$ converges.

If (f_n) converges pointwise on S, then we define f: $S \to \mathbf{R}$ by
$f(x) = \lim_{n\to\infty} f_n(x)$ for each x in S and we say that (f_n)converges to f pointwise on S. **(Def. 6.1.1, Lebl)**

Example. Let $f_n : [0, 1] \to \mathbf{R}$ where $f_n(x) = x^n$ for each positive integer n.
For each x in $[0, 1)$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$, and $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$.

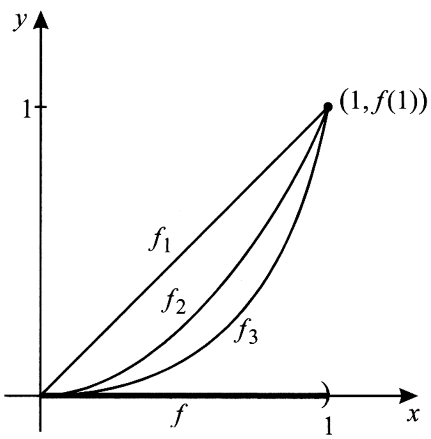

Figure 35.1 $f_n(x) = x^n$ for $x \in [0,1]$

So, (f_n) converges pointwise to the function $f(x) = \begin{cases} 0, & if\ 0 \le x < 1 \\ 1, & if\ x = 1 \end{cases}$

Note that although each f_n is continuous (and differentiable) on [0, 1], the limit function f is discontinuous (and not differentiable) at $x = 1$.

SEE VIDEO and Interactive Version of this example at
http://sandsduchon.org/sands/MATH301/Videos/Chapter9/MATH301Chapter9Videos.html

Example. For *x* in [0, 1] and $n \geq 2$, let

$$f_n(x) = \begin{cases} n^2 x, & if\ 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \frac{2}{n}\right), & if\ \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0, & if\ \frac{2}{n} < x \leq 1 \end{cases}$$

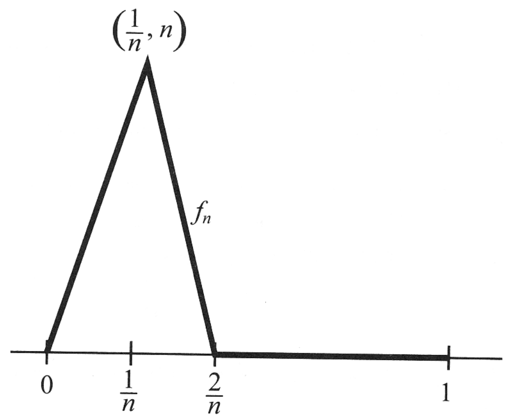

Figure 35.2 Example 35.3

SEE Interactive Version of this example at
http://sandsduchon.org/sands/MATH301/Videos/Chapter9/MATH301Chapter9Videos.html

Then $\lim_{n\to\infty} f_n(x) = 0$ so the limit function is *f*(*x*) = 0.

Note that $\int_0^1 f_n(x)dx = 1$ but $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 f(x)dx = 0$
so we have

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

Example. For *x* in [0, 2π] and positive integers *n*, let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$.
Then (f_n) converges to $f(x) = 0$. SEE VIDEO and Interactive Version of this example at
http://sandsduchon.org/sands/MATH301/Videos/Chapter9/MATH301Chapter9Videos.html

However, it turns out that the sequence of derivatives (f_n') does not converge to f'.
In fact, the sequence (f_n') does not converge for any *x*.

Example. For *x* in [0, 1] and positive integers *n*, let $f_n(x) = 2x + \frac{x}{n}$.
Then (f_n) converges to $f(x) = 2x$.

It turns out that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$ and also
it turns out that the sequence of derivatives (f_n') converges to f'.

(Def. 6.1.6, Lebl) **Definition**. let (f_n) be a sequence of functions defined on a subset S of **R**. Then (f_n) **converges uniformly** on S to a function f defined on S if

for each $\varepsilon > 0$ there exists a number N such that for all x in S and all $n > N$, $|f_n(x) - f(x)| < \varepsilon$.

To say that a sequence (f_n) converges uniformly on S is to say that there exists a function f to which (f_n) converges uniformly on S.

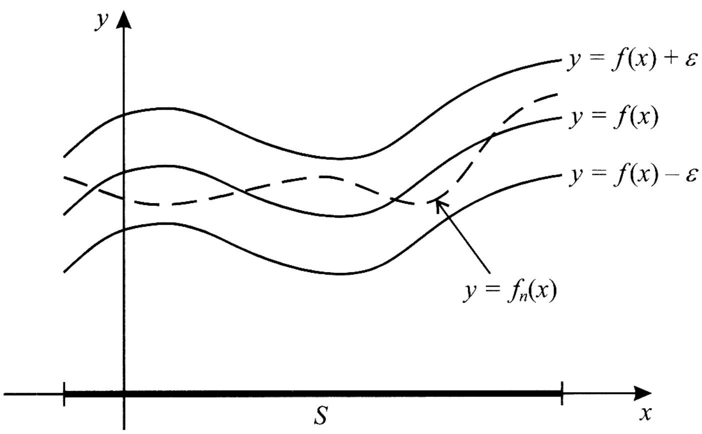

Figure 35.3 Uniform convergence on S

Example. Let $f_n : [0, 1] \to \mathbf{R}$ where $f_n(x) = x^n$ for each positive integer n. (same sequence as in earlier example) It turns out that although (f_n) converges pointwise to f, the sequence (f_n) does not converge uniformly to f.

Discussion: (f_n) **converges uniformly** on [0,1] if there is a function f satisfying:

For each $\varepsilon > 0$ there exists a number N such that for all x in [0,1], $n > N$ implies that $|f_n(x) - f(x)| < \varepsilon$.

NOT converging uniformly on [0, 1] to f means:

$\exists\, \varepsilon > 0$ such that for all N, $\exists\, x$ in [0,1] and $n > N$ for which $|x^n - f(x)| \geq \varepsilon$.

Proof that the sequence (f_n) does not converge uniformly to f:

Pick $\varepsilon = 1/2$ and say that x in [0, 1), so $f(x) = 0$ and $|x^n - f(x)| = x^n$.

We want $x^n \geq 1/2$.

So, $x \geq \sqrt[n]{\frac{1}{2}} = (1/2)^{1/n} = 2^{-1/n}$

Thus for $\varepsilon = 1/2$, given any N, for n > N, let $x = 2^{-1/n}$. Then $x^n > \left(2^{-1/n}\right)^n = 1/2$

SEE VIDEO and Interactive Version of this example at
http://sandsduchon.org/sands/MATH301/Videos/Chapter9/MATH301Chapter9Videos.html

Example. The earlier example with the sequence (f_n) depicted in figure 35.2, the convergence to f is not uniform.

Example. For x in [0, 2π] and positive integers n, let $f_n(x) = \frac{\sin nx}{\sqrt{n}}$.

Then (f_n) converges uniformly to $f(x) = 0$ on the interval [0, 2π].
SEE VIDEO of proof at http://sandsduchon.org/sands/MATH301/Videos/Chapter9/MATH301Chapter9Videos.html

Cauchy criterion for uniform convergence of sequences of functions (Prop. 6.1.13, Lebl)

Theorem. let (f_n) be a sequence of functions defined on a subset S of **R**.
There exists a function f such that (f_n) converges to f uniformly on S
iff
the Cauchy criterion is satisfied:
for every ε > 0, there exists a number N such that $| f_n(x) - f_m(x) | < \varepsilon$ for all x in S and all $m, n > N$.

Theorem.** Let (f_n) be a sequence of continuous functions defined on a set S and suppose that (f_n) **converges uniformly on *S to a function $f: S \rightarrow$ **R**. Then f is continuous on S. **(Prop. 6.2.2, Lebl)**

Example: Let $f_n : [0, 1] \rightarrow$ **R** where $f_n(x) = x^n$ for each positive integer n.

The sequence (f_n) converges to $f(x) = \begin{cases} 0, & if\ 0 \le x < 1 \\ 1, & if\ x = 1 \end{cases}$

but the sequence (f_n) cannot converge *uniformly* to f. Otherwise, by the theorem*, the limit f must be continuous on [0, 1], but clearly in this example, f is discontinuous at $x = 1$.

Theorem. Let (f_n) be a sequence of continuous functions defined on an interval [a, b] and suppose that (f_n) converges uniformly on [a, b] to a function f. Then

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Theorem. Suppose that (f_n) converges to f on an interval [a, b]. Suppose also that each (f_n') exists and is continuous on [a, b], and the sequence (f_n') converges uniformly on [a, b].

Then $\lim_{n\to\infty} f_n'(x) = f'(x)$ for each x in [a, b].

Definition. If $(f_n)_{n=0}^{\infty}$ is a sequence of functions defined on a set S, the series $\sum_{n=0}^{\infty} f_n$ is said to converge pointwise on S
iff the sequence of partial sums given by $s_n(x) = \sum_{k=0}^{n} f_k(x)$ converges pointwise on S.
(A similar definition applies to uniform convergence.)

Theorem. **Weierstrass M-Test**. Suppose that (f_n) is a sequence of functions defined on S and (M_n) is a sequence of nonnegative numbers such that $|f_n(x)| \leq M_n$ for all x in S and all positive integers n.
If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on S. **(Th. 3, Chapter 4, Sec. 13, Zakon, p.240)**

Example. Consider the series $\sum \left(\frac{\sin x}{2}\right)^n$ for x in **R**.

$\left|\left(\frac{\sin x}{2}\right)^n\right| = \frac{|\sin x|^n}{2^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$ and $\sum \left(\frac{1}{2}\right)^n$ converges (geometric series with r =1/2)

so $\sum \left(\frac{\sin x}{2}\right)^n$ converges by the Weierstrass M-Test, with $M_n = \left(\frac{1}{2}\right)^n$

Example. Consider the series $\sum_{n=0}^{\infty} f_n$ where $f_n(x) = \frac{x^n}{n!}$.
It turns out that the series converges pointwise on **R** but does not converge uniformly on **R**.
However, the series converges uniformly on any closed interval [-t, t].

Theorem. Let $\sum_{n=0}^{\infty} f_n$ be a series of functions defined on a set S. Suppose that each f_n is continuous on S and that the series converges uniformly to a function f on S. Then $f = \sum_{n=0}^{\infty} f_n$ is continuous on S.

Theorem. Let $\sum_{n=0}^{\infty} f_n$ be a series of functions defined on an interval [a, b]. Suppose that each f_n is continuous on [a, b] and that the series converges uniformly to a function f on [a, b].
Then $\int_a^b f(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx$.

Example. The geometric series $\sum_{n=0}^{\infty} (-t)^n = \frac{1}{1+t}$ for t in the interval (-1, 1).

It can be shown by the Weierstrass M-test that the series converges uniformly on any interval [-r, r] contained in (-1, 1). According to the theorem above, if x is in (-1, 1), we can integrate term by term and get

$$\int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} \int_0^x (-t)^n dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

so,

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ ...$$

Theorem. Let $\sum_{n=0}^{\infty} f_n$ be a series of functions that converges to a function f on an interval [a, b]. Suppose that for each n, f_n' exists and is continuous on [a, b] and that the series of derivatives $\sum_{n=0}^{\infty} f_n'$ is uniformly convergent on [a, b]. Then $f'(x) = \sum_{n=0}^{\infty} f_n'(x)$ for all x in [a, b].