Infinite Series (Week 6)

Convergence of Infinite Series

Summation notation

$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n$ Def. 2.5.1 Lebl, p. 72

Let (s_n) be the sequence of **partial sums** defined by $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$.

If (s_n) converges to a real number s, we say that the infinite series $\sum_{n=1}^{\infty} a_n$ is **convergent** and we write $\sum_{n=1}^{\infty} a_n = s$.

s is called the **sum** of the series. A series that is not convergent is called **divergent**. If lim s_n = +∞ we say that the series **diverges to +∞** and we write $\sum_{n=1}^{\infty} a_n = +\infty$.

Example: Consider the infinite series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

Partial sum: $s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$

By induction, it can be shown that $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for every $n \in$ **N.**
Therefore, $\lim s_n = \lim \frac{n}{n+1} = 1$.
So, the infinite series converges to 1, and we write $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.

An alternative approach:

Thinking of partial fractions from calculus, note that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

So, $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right]$

and $s_n = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$ (called a **telescoping sum**)

$= 1 - \frac{1}{n+1}$

Therefore, $\lim s_n = \lim \left(1 - \frac{1}{n+1}\right) = 1$, just as with our other approach.

Example: The **harmonic series** $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$.
(Shown in Sequence notes, week 4, page 7)

Theorem. Suppose that $\sum a_n = s$ and $\sum b_n = t$. **(Prop. 2.5.10, Lebl, page 75)**

Then $\sum(a_n + b_n) = s + t$, and $\sum(ka_n) = ks$ for every $k \in \mathbf{R}$.

Theorem. If $\sum a_n$ is a convergent series, then $\lim a_n = 0$. **(Prop. 2.5.8, Lebl)**

Proof:
If $\sum a_n$ converges, then the sequence of partial sums (s_n) must have a finite limit. Call it s.

Note that $s_n - s_{n-1} = a_n$.

So, $\lim (s_n - s_{n-1}) = \lim a_n$.
$\lim s_n - \lim s_{n-1} = \lim a_n$.
$s - s = \lim a_n$.
$0 = \lim a_n$.

I have specifically stated the theorem's related result which is familiar from calculus:
Corollary (nth Term Test): If $\lim a_n \neq 0$, then $\sum a_n$ diverges.

Note that the corollary is the contrapositive of the theorem and thus follows directly from the theorem.

Example: $\sum_{n=1}^{\infty} \frac{n}{10n+4}$ diverges, because $\lim \frac{n}{10n+4} = \frac{1}{10} \neq 0$.

Theorem. (Cauchy Criterion for Series) **(Prop. 2.5.7, Lebl)**
The infinite series $\sum a_n$ converges
iff
for each $\varepsilon > 0$ there exists a number N such that, if $n \geq m > N$, then $| a_m + a_{m+1} + \ldots + a_n | < \varepsilon$.

Proof:
Suppose that $\sum a_n$ converges. Then the sequence (s_n) of partial sums converges, and so (s_n) must be a Cauchy sequence.

Given $\varepsilon > 0$, there exists N such that m, n > N implies that $| s_n - s_m | < \varepsilon$.
So if $n \geq m > N + 1$, then $m - 1 > N$, so that $| s_n - s_{m-1} | < \varepsilon$.

But $|s_n - s_{m-1}| = |(a_1 + a_2 + \ldots + a_n) - (a_1 + a_2 + \ldots + a_{m-1})| = | a_m + a_{m+1} + \ldots + a_n|$
so $| a_m + a_{m+1} + \ldots + a_n | < \varepsilon$ as desired.

Conversely, suppose that for each $\varepsilon > 0$ there exists a number N such that
$n \geq m > N$ implies that $| a_m + a_{m+1} + \ldots + a_n | < \varepsilon$.

Then for n > m > N, we have m + 1 > N so that $|s_n - s_m| = | a_{m+1} + a_{m+2} + \ldots + a_n | < \varepsilon$.

Therefore, the sequence of partial sums (s_n) must be a Cauchy sequence and therefore converges.
Since the sequence of partial sums converges, $\sum a_n$ converges.

Geometric Series $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots$

Partial sum: $s_n = 1 + r + r^2 + \cdots + r^n$
It can be proven by induction that

$$1 + r + r^2 + \cdots + r^n = \frac{1-r^{n+1}}{1-r} \text{ for every } n \in \mathbf{N}, r \neq 1.$$

Another result from our week 3 study of sequences showed that $\lim_{n\to\infty} x^n = 0$ iff $|x| < 1$.
So, $\lim_{n\to\infty} r^n = 0$ provided $|r| < 1$.

If $|r| < 1$, $\lim s_n = \lim \frac{1-r^{n+1}}{1-r} = \frac{1-\lim r^{n+1}}{1-r} = \frac{1-r\lim r^n}{1-r} = \frac{1-r(0)}{1-r} = \frac{1}{1-r}$

If $r = 1$, then $\sum_{n=0}^{\infty} 1^n = 1 + 1 + 1 + \cdots$, which diverges to $+\infty$.

If $r = -1$, then $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + 1 - \cdots$.
Note that if n is even, $s_n = 1$ but if n is odd, $s_n = 0$. Therefore the sequence (s_n) of partial sums does not converge, and so $\sum_{n=0}^{\infty} (-1)^n$ diverges. Alternatively, $\lim (-1)^n$ does not exist, so the series diverges.

If $|r| > 1$, $\lim a_n = \lim r^n \neq 0$, so the geometric series diverges.

Summary:
The **geometric series** $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ only if $|r| < 1$.
For $|r| \geq 1$, the series diverges.

Example: ***p*-Series** $\sum \frac{1}{n^p}$. The p-series converges if $p > 1$ and diverges if $p \leq 1$. **(Prop. 2.5.15, Lebl)**

Convergence Tests

Theorem (Comparison Test) **(Def. 2.5.14, Lebl)**
Let $\sum a_n$ and $\sum b_n$ be infinite series of nonnegative terms. Then

(a) If $\sum a_n$ converges and $0 \leq b_n \leq a_n$ for all n, then $\sum b_n$ converges.

(b) If $\sum a_n = +\infty$ and $0 \leq a_n \leq b_n$ for all n, then $\sum b_n = +\infty$.

Examples:

$\sum \frac{1}{(n+1)^2}$. Since $0 < \frac{1}{(n+1)^2} < \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ is a convergent p-series with $p = 2$, the given series converges by the comparison test.

$\sum \frac{1}{n-\sqrt{2}}$. Since $0 < \frac{1}{n} < \frac{1}{n-\sqrt{2}}$ (for $n > 2$) and the harmonic series $\sum \frac{1}{n}$ diverges, the given series diverges by the comparison test.

Definition. If $\sum|a_n|$ converges, then the series $\sum a_n$ converges is said to **converge absolutely** (or to be **absolutely convergent**). If $\sum a_n$ converges but $\sum|a_n|$ diverges, then $\sum a_n$ is said to **converge conditionally** (or be **conditionally convergent**). **(Def. 2.5.12, Lebl)**

Theorem: If a series converges absolutely, then it converges. **(Prop. 2.5.13, Lebl)**

Example: $\sum \frac{(-1)^n}{n^2}$ converges absolutely. (Series of absolute values is a *p*-series with *p* = 2).

Theorem (Ratio Test) Let $\sum a_n$ be an infinite series of nonzero terms. **(Prop. 2.5.17, Lebl)**

Suppose $\lim\left|\frac{a_{n+1}}{a_n}\right|$ exists and is equal to L.

(a) If $L < 1$, then the series converges absolutely.
(b) If $L > 1$, the series diverges.
(c) Otherwise, the test gives no information about convergence or divergence.

Theorem (Root Test) Given a series $\sum a_n$, let $\alpha = \limsup|a_n|^{1/n}$. **(Prop. 2.6.1, Lebl)**

(a) If $\alpha < 1$, then the series converges absolutely.
(b) If $\alpha > 1$, the series diverges.
(c) Otherwise $\alpha = 1$, and the test gives no information about convergence or divergence.

When applying the root test, it can be handy to recall that $\lim n^{1/n} = 1$.

Examples

$\sum \frac{n}{2^n}$ converges (can apply the ratio test or the root test).

$$\lim\left|\frac{a_{n+1}}{a_n}\right| = \lim\frac{(n+1)}{2^{n+1}}\cdot\frac{2^n}{n} = \lim\frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2} < 1.$$

$$\limsup|a_n|^{1/n} = \limsup\frac{n^{1/n}}{(2^n)^{1/n}} = \limsup\frac{n^{1/n}}{2} = \frac{1}{2}\lim n^{1/n} = \frac{1}{2} < 1$$

$\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\cdots$ converges (compare to the convergent geometric series with *r* = 1/2.)

$\sum\frac{2^n}{n!}$ converges (can apply the ratio test).