Differentiation **(notes by S. Sands)**

The Derivative

(Def. 4.1.1, Lebl, p. 131)

Definition: Let $f\colon I \to \mathbf{R}$ be a real-valued function defined on an interval I containing the point c. (We allow the possibility that c is an endpoint of the interval.) We say that f is **differentiable** at c (or has a **derivative** at c) if the limit

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$$

exists and is finite. We call the limit the **derivative of f at c**, denoted $f'(c)$, so

$$f'(c)=\frac{f(x)-f(c)}{x-c}$$

whenever the limit exists and is finite.

If the function f is differentiable at each point of the set $S \subseteq I$, then f is said to be **differentiable on S**, and the function $f'\colon S \to \mathbf{R}$ is called the derivative of f on S.

Examples: For relatively simple functions, such as low degree polynomials, some powers of x, and simple piecewise-defined functions, it is relatively easy to find the derivative using the limit definition.

Example: Let $f(x)=\begin{cases} x\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$. f is not differentiable at $x = 0$ because the limit

$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{x\sin\frac{1}{x}-0}{x}=\lim_{x\to 0}\sin\frac{1}{x}$ does not exist.

Geometrically, the difference quotient $\frac{f(x)-f(c)}{x-c}$ represents the slope of the secant line through the points $(c, f(c))$ and $(x, f(x))$. As x approaches c, this ratio approaches the slope of the tangent line at c.

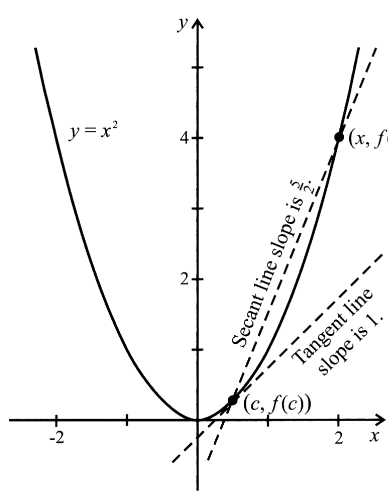

Figure 25.1 $f(x) = x^2$ with $c = 1/2$ and $x = 2$

Sequential Condition for Derivatives: Let I be an interval containing the point c and $f: I \to \mathbf{R}$. Then f is differentiable at c
iff for every sequence (x_n) in I that converges to c with $x_n \neq c$ for all n, the sequence $\left(\frac{f(x_n)-f(c)}{x_n-c}\right)$ converges.
Furthermore, if f is differentiable at c, then the sequence of quotients converges to $f'(c)$.

Example: Using the sequential condition to show that a function is not differentiable at a point:

Let $f(x) = |x|$ and let $x_n = (-1)^n/n$ for positive integers n. Then the sequence (x_n) converges to 0.
The corresponding sequence of quotients has terms $\frac{f(x_n)-f(c)}{x_n-c} = \frac{\left|\frac{(-1)^n}{n}\right|-0}{\frac{(-1)^n}{n}-0}$.
For n even, the quotient is 1 but for n odd, the quotient is -1, so the sequence of quotients oscillates between -1 and 1, and so does not converge.

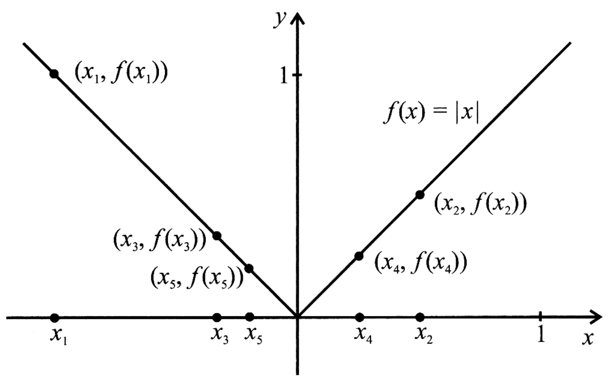

Figure 25.2 $f(x) = |x|$ and $x_n = (-1)^n/n$

REMARK: $f(x) = |x|$ is continuous on **R** but not differentiable at 0.

(Prop. 4.1.4, Lebl)
Theorem. If $f: I \to \mathbf{R}$ is **differentiable at a point c** in I, then f is **continuous at c**.

Differentiation Rules **(Prop. 4.1.5, 4.16, 4.17, Lebl)**
Theorem. Suppose that $f: I \to \mathbf{R}$ and $f: I \to \mathbf{R}$ are differentiable at c in I. Then

(a) if $k \in \mathbf{R}$, then the function kf is differentiable at c and $(kf)'(c) = k f'(c)$.

(b) **Sum Rule**: The function $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$.

(c) **Product Rule**: The function fg is differentiable at c and $(fg)'(c) = f(c)\, g'(c) + g(c)\, f'(c)$.

(d) **Quotient Rule**: If $g(x) \neq 0$, then the function f/g is differentiable at c and

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$$

Power Rules. (Integer exponents)
For any positive integer n, if $f(x) = x^n$ for all x in **R**, then $f'(x) = nx^{n-1}$ for all x in **R**.

For any negative integer n, if $f(x) = x^n$ for all $x \neq 0$, then $f'(x) = nx^{n-1}$ for all $x \neq 0$.

(Prop. 4.1.8, Lebl, p. 132)
Theorem. **Chain Rule**.
Let I and J be intervals in **R**, let $f: I \to \mathbf{R}$ and $g: I \to \mathbf{R}$, where $f(I) \subseteq J$, and let c in I.

If f is differentiable at c and g is differentiable at $f(c)$,
then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))\, f'(c)$.

Examples:

(1) Differentiate $h(x) = \sin(1/x)$ for $x \neq 0$.

$h(x) = \sin(1/x) = (g \circ f)(x)$ where $f(x) = 1/x = x^{-1}$ and $g(x) = \sin x$.

Since $f'(x) = -x^{-2} = -1/x^2$ (power rule) and $g'(x) = \cos x$ (assumed),

applying the chain rule, $h'(x) = g'(f(x))\, f'(x) = [\cos(1/x)]\,(-1/x^2) = (-1/x^2)\cos(1/x)$.

(2) Differentiate $r(x) = x \sin(1/x)$ for $x \neq 0$.

Note that $r(x) = x\, h(x)$. Applying the product rule,
$r'(x) = x\, h'(x) + 1\, h(x) = x\,(-1/x^2)\cos(1/x) + \sin(1/x) = (-1/x)\cos(1/x) + \sin(1/x)$.

(3) There is a continuous function on **R** that has a derivative at all but one real number.

$f(x) = |x|$ qualifies (see earlier example on previous page).

Another example:

Let $r(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ This is an extension of the function $r(x)$ in part (2).

The derivative has been found for all $x \neq 0$.
When $x = 0$, the derivative does not exist. (See the second example on the first page.)

Calculus Theorems

(Th. 4.2.2, Lebl) Theorem: If f is differentiable on an open interval (a, b) and if f assumes its maximum or minimum at a point $c \in (a, b)$, then $f'(c) = 0$.

Remark: Recall that in calculus when f continuous on closed interval $[a, b]$, to find the maximum and minimum of the function, we look at the y-values corresponding to:
(1) points c where $f'(c) = 0$, (2) endpoints a and b, and (3) points c where $f'(c)$ does not exist.

(Th. 4.2.3, Lebl)
Rolle's Theorem. Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

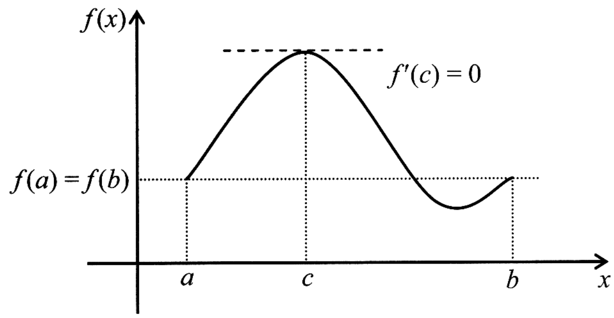

Figure 26.1 Rolle's theorem

The Mean Value Theorem is an extension of Rolle's Theorem where the endpoints do not have the same y-values.

(Th. 4.2.4, Lebl)
Mean Value Theorem.
Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

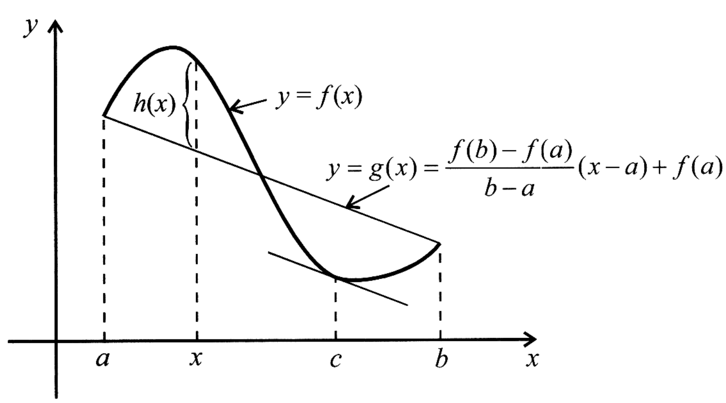

Figure 26.2 The mean value theorem

Examples:

- The Mean Value Theorem can be used to prove Bernoulli's inequality:
  $(1 + x)^n \geq 1 + nx$, for $x > -1$, and all n in $\mathbf{N}$.

- The Mean Value Theorem can be used to estimate the value of a function near a point.

For instance, $\sqrt{40}$ can be approximated using the Mean Value Theorem and the knowledge that 40 is relatively close to 36, a perfect square.

Additional familiar calculus concepts can be established.

(Prop. 4.2.5, Lebl)

Theorem. Let f be a function which is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on (a, b).

Corollary. Let f and g be continuous on $[a, b]$ and differentiable on (a, b).
Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$.
Then there exists a constant C such that $f = g + C$ on $[a, b]$.

(Prop. 4.2.6, Lebl)

Theorem. Let f be differentiable on an interval I. Then
(a) if $f'(x) > 0$ for all $x \in I$, then f is strictly increasing on I, and
(b) if $f'(x) < 0$ for all $x \in I$, then f is strictly decreasing on I.

(Darboux Th. 4.2.9, Lebl)

Intermediate Value Theorem for Derivatives.
Let f be differentiable on $[a, b]$ and suppose that k is a number between $f'(a)$ and $f'(b)$. Then there exists a point $c \in (a, b)$ such that $f'(c) = k$.

(Th. 4.4.2, Lebl)

Inverse Function Theorem. Suppose f is differentiable on an interval I and $f'(x) \neq 0$ for all $x \in I$. Then f is injective, f^{-1} is differentiable on $f(I)$, and

$$(f^{-1})'(y) = \frac{1}{f'(x)} \qquad \text{where } y = f(x).$$

or, equivalently, $f'(x) = \dfrac{1}{(f^{-1})'(y)}$ where $y = f(x)$.

or, equivalently, $f'(x) = \dfrac{1}{g'(y)}$ where $y = f(x)$ and $g = f^{-1}$.

(Cor. 4.4.3, Lebl)

Power Rule (Power of form 1/n)

For any positive integer n, if $f(x) = x^{1/n}$ for all $x > 0$, then $f'(x) = (1/n)\, x^{1/n - 1}$ for $x > 0$.

Proof:
Since f is injective, the inverse function exists.

To apply the Inverse Function Theorem, we need to know f^{-1}, the inverse function.

$y = f(x) = x^{1/n}$ Now raise both sides to the nth power.

so $y^n = x$, and so $f^{-1}(y) = y^n$.

$g(y) = y^n$ is the inverse function.

$g'(y) = n\, y^{n-1}$ by the earlier power rule for positive integer powers.

Applying the Inverse Function Theorem,

$$f'(x) = \frac{1}{g'(y)} = \frac{1}{ny^{n-1}} = \frac{1}{n}\, y^{-n+1} \quad \text{where } y = f(x) = x^{1/n}$$

Substituting for y, $\frac{1}{n}\, y^{-n+1} = \frac{1}{n}\left(x^{1/n}\right)^{-n+1} = \frac{1}{n} x^{-n\left(\frac{1}{n}\right)+1/n} = \frac{1}{n} x^{-1+1/n} = \frac{1}{n} x^{\frac{1}{n}-1}$

So, $f'(x) = (1/n)\, x^{1/n - 1}$

Power Rule (Rational Exponents)
For any nonzero integers m and n, if $f(x) = x^{m/n}$ for all $x > 0$, then $f'(x) = (m/n)\, x^{m/n - 1}$ for $x > 0$.

To establish this, use the fact that $f(x) = x^{m/n} = (x^{1/n})^m$ and apply the Chain Rule and the previous example.

L'Hospital's Rule

Cauchy Mean Value Theorem: Let f and g be functions that are continuous on $[a, b]$ and differentiable (a, b). Then there exists at least one point $c \in (a, b)$ such that

$[f(b) - f(a)]\, g'(c) = [g(b) - g(a)]\, f'(c)$.

Indeterminate forms: $0/0$, ∞/∞, $0 \cdot \infty$, 1^∞, ∞^0, 0^0, $\infty - \infty$.

(Th. 2.4.1, TRENCH) L'Hospital's Rule for the indeterminate form 0/0.
Let f and g be functions that are continuous on $[a, b]$ and differentiable (a, b).
Suppose that $c \in [a, b]$ and that $f(c) = g(c) = 0$. Suppose also that $g'(c) \neq 0$ for $x \in U$, where U is the intersection of (a, b) and some deleted neighborhood of c.

If $\lim_{x\to c} \frac{f'(x)}{g'(x)} = L$, with $L \in \mathbf{R}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = L$.

Examples (indeterminate form 0/0):

$$\lim_{x\to 0} \frac{\sin x}{x} = 1, \quad \lim_{x\to 1} \frac{2x^2 - 3x + 1}{x - 1} = 1, \quad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}, \quad \lim_{x\to 0} \frac{e^{2x}-1}{x} = 2$$

Limits at Infinity

Definition. Let f: $(b, \infty) \to \mathbf{R}$ where $b \in \mathbf{R}$. We say that $L \in \mathbf{R}$ is the limit of f as $x \to \infty$, written

$$\lim_{x\to\infty} f(x) = L$$

provided that
for each $\varepsilon > 0$ there exists a real number $N > b$ such that $x > N$ implies that $|f(x) - L| < \varepsilon$.

Definition. Let f: $(b, \infty) \to \mathbf{R}$ where $b \in \mathbf{R}$. We say that f tends to ∞ as $x \to \infty$, written

$$\lim_{x\to\infty} f(x) = \infty$$

provided that
for each $\alpha \in \mathbf{R}$ there exists a real number $N > b$ such that $x > N$ implies that $f(x) > \alpha$.

(Th. 2.4.1, TRENCH) L'Hospital's Rule for the indeterminate form ∞/∞.
Let f and g be differentiable on (b, ∞).
Suppose that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and that $g'(x) \neq 0$ for $x \in (b, \infty)$.

If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$, with $L \in \mathbf{R}$, then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Examples (indeterminate forms ∞/∞, $0 \cdot \infty$, 0^0)

$$\lim_{x\to\infty} \frac{\ln x}{x} = 0, \qquad \lim_{x\to 0+} (x)(-\ln x) = 0, \qquad \lim_{x\to 0+} x^x = 1$$