March 15, 2017

Lecture 20
Numerical integration: Newton-Cotes quadrature rules

By the end of this lecture, students will be able to

- use Newton-Cotes quadrature rules.

1 Introduction

In this lecture, we introduce numerical methods *via quadrature rules* for approximating the integral

$$I(f) = \int_a^b f(x)\,dx,$$

where f is an arbitrary continuous function in $[a, b]$.

A m-point *quadrature rule* Q for the above integral is an approximation of the form

$$I_Q(f) = \sum_{k=1}^{m} w_k f(x_k).$$

The x_k are the *abscissas* and the w_k are the *weights*. The abscissas and weights define the rule and are chosen so that $I_Q(f) \approx I(f)$.

We start by presenting the *Netwon-Cotes* family of quadrature rules. These rules are derived by integrating a polynomial interpolant of the integrand $f(x)$.

2 The Newton-Cotes Rules

One way to derive a quadrature rule Q (consists of abscissas and weights) is to integrate a polynomial approximation $p(x)$ of the integrand $f(x)$. The philosophy is that $p(x) \approx f(x)$ implies

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx.$$

See Figure 1.

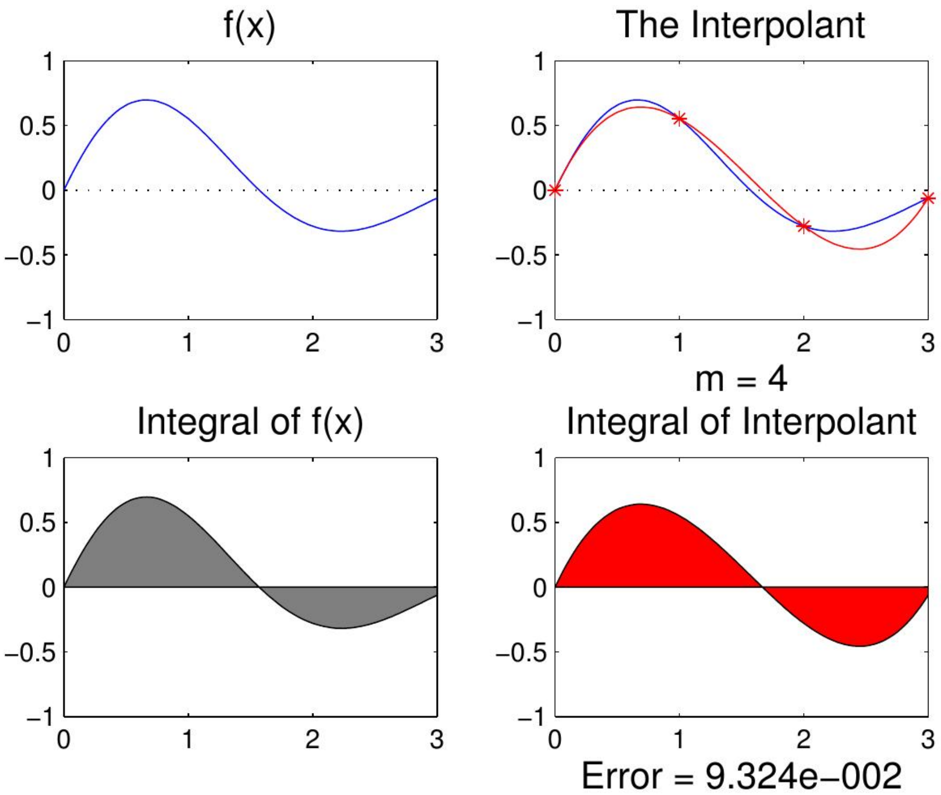

Figure 1: The Newton-Cotes idea

2.1 Closed Newton-Cotes rules

The *closed Newton-Cotes* quadrature rules are obtained by integrating uniformly spaced polynomial interpolants of the integrand. The m-point closed Newton-Cotes rule ($m \geqslant 2$) is defined by

$$Q_{NC(m)} = \int_a^b p_{m-1}(x)\,dx,$$

where $p_{m-1}(x) = \sum_{k=0}^{m-1} f(x_k)\phi_k(x)$ is the equispaced Lagrange interpolant of degree $m-1$ for $f(x)$. Here $x_k = a + \frac{k}{m-1}(b-a)$, $k = 0, \cdots, m-1$ are the equispaced nodes, and $\phi_k(x)$ is the k-th Lagrange characteristic polynomial of degree $m-1$. Hence,

$$\begin{aligned} Q_{NC(m)} &= \int_a^b \sum_{k=0}^{m-1} f(x_k)\phi_k(x)\,dx \\ &= \sum_{k=0}^{m-1} f(x_k) \int_a^b \phi_k(x)\,dx. \end{aligned}$$

We have transformed the problem of finding the quadrature weights of the Newton-Cotes rules to the evaluation of integrals for Lagrange characteristic polynomials. [1]

✍ Find the 2-point closed Newton-Cotes rule (*trapezoidal rule*), the 3-point closed Newton-Cotes rule (*Simpson rule*), and the 4-point closed Newton-Cotes rule (*Simpson 3/8 rule*).
File: `NewtonCotesRules.m` (symbolic integration).

[1] Since the two endpoints $x_0 = a$ and $x_{m-1} = b$ are included in the abscissas, the quadrature rules are referred as *closed* Netwon-Cotes rules.

If $m = 2$, we obtain the *trapezoidal rule*:

$$\begin{aligned} Q_{NC(2)} &= \int_a^b f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}\,dx \\ &= (b-a)\left(\frac{1}{2}f(a) + \frac{1}{2}f(b)\right). \end{aligned}$$

If $m = 3$, setting $c = a + (b-a)/2$ we obtain the *Simpson rule*:

$$\begin{aligned} Q_{NC(3)} &= \int_a^b f(a)\frac{(x-c)(x-b)}{(a-c)(a-b)} + f(c)\frac{(x-a)(x-b)}{(c-a)(c-b)} + f(b)\frac{(x-a)(x-c)}{(b-a)(b-c)}\,dx \\ &= \frac{b-a}{6}\left(f(a) + 4f(c) + f(b)\right) \\ &= \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right). \end{aligned}$$

If $m = 4$, we obtain the *Simpson 3/8 rule* (we used Matlab's symbolic integration to obtain the weights):

$$Q_{NC(4)} = \frac{b-a}{8}\left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right).$$

2.2 Open Newton-Cotes rules

The open Newton-Cotes rules also use equispaced nodes as abscissas, but do not include the two endpoints. The abscissas for a m-point open Newton-Cotes rule are

$$x_k = a + \frac{k}{m+1}(b-a), \quad k = 1, \cdots, m.$$

The quadrature formulation are the same as that for the closed Newton-Cotes rules, with a different set of Lagrange characteristic polynomials.

If $m = 1$, we obtain the *midpoint rule*:

$$\begin{aligned} Q_{ONC(1)} &= \int_a^b f(\frac{a+b}{2})\,dx \\ &= (b-a)f\left(\frac{a+b}{2}\right). \end{aligned}$$

If $m = 2$, we obtain the *Trapezoid method*:

$$Q_{ONC(2)} = \frac{(b-a)}{2}\left(f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right).$$