3.

Sally	Math
Ruth	Physics
Sam	Econ

4.

a	a
b	b

In Exercises 5–8, write the relation as a table.

5. $R = \{(a, 6), (b, 2), (a, 1), (c, 1)\}$

6. $R =$ {(Roger, Music), (Pat, History), (Ben, Math), (Pat, PolySci)}

7. The relation R on $\{1, 2, 3, 4\}$ defined by $(x, y) \in R$ if $x^2 \geq y$

8. The relation R from the set X of planets to the set Y of integers defined by $(x, y) \in R$ if x is in position y from the sun (nearest the sun being in position 1, second nearest the sun being in position 2, and so on)

In Exercises 9–12, draw the digraph of the relation.

9. The relation of Exercise 4 on $\{a, b, c\}$

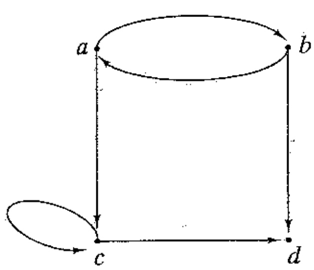
10. The relation $R = \{(1, 2), (2, 1), (3, 3), (1, 1), (2, 2)\}$ on $X = \{1, 2, 3\}$

11. The relation $R = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$ on $\{1, 2, 3, 4\}$

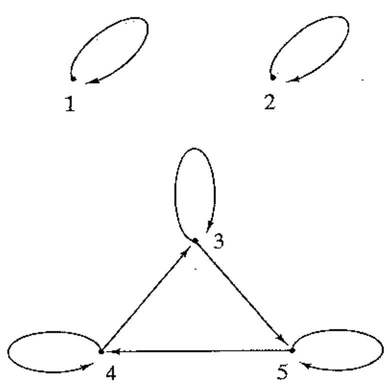
12. The relation of Exercise 7

In Exercises 13–16, write the relation as a set of ordered pairs.

13.

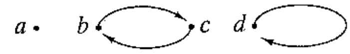


14.



15.

16.



17. Find the inverse (as a set of ordered pairs) of each relation in Exercises 1–16.

Exercises 18 and 19 refer to the relation R on the set $\{1, 2, 3, 4, 5\}$ defined by the rule $(x, y) \in R$ if 3 divides $x - y$.

18. List the elements of R.

19. List the elements of R^{-1}.

20. Repeat Exercises 18 and 19 for the relation R on the set $\{1, 2, 3, 4, 5\}$ defined by the rule $(x, y) \in R$ if $x + y \leq 6$.

21. Repeat Exercises 18 and 19 for the relation R on the set $\{1, 2, 3, 4, 5\}$ defined by the rule $(x, y) \in R$ if $x = y - 1$.

22. Is the relation of Exercise 20 reflexive, symmetric, antisymmetric, transitive, and/or a partial order?

23. Is the relation of Exercise 21 reflexive, symmetric, antisymmetric, transitive, and/or a partial order?

In Exercises 24–31, determine whether each relation defined on the set of positive integers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order.

24. $(x, y) \in R$ if $x = y^2$.

25. $(x, y) \in R$ if $x > y$.

26. $(x, y) \in R$ if $x \geq y$.

27. $(x, y) \in R$ if $x = y$.

28. $(x, y) \in R$ if 3 divides $x - y$.

29. $(x, y) \in R$ if 3 divides $x + 2y$.

30. $(x, y) \in R$ if $x - y = 2$.

31. $(x, y) \in R$ if $|x - y| = 2$.

32. Let X be a nonempty set. Define a relation on $\mathcal{P}(X)$, the power set of X, as $(A, B) \in R$ if $A \subseteq B$. Is this relation reflexive, symmetric, antisymmetric, transitive, and/or a partial order?

33. Prove that a relation R on a set X is antisymmetric if and only if for all $x, y \in X$, if $(x, y) \in R$ and $x \neq y$, then $(y, x) \notin R$.

34. Let X be the set of all four-bit strings (e.g., 0011, 0101, 1000). Define a relation R on X as $s_1 \, R \, s_2$ if some substring of s_1 of length 2 is equal to some substring of s_2 of length 2. Examples: 0111 R 1010 (because both 0111 and 1010 contain 01). 1110 $\not R$ 0001 (because 1110 and 0001 do not share a common substring of length 2). Is this relation reflexive, symmetric, antisymmetric, transitive, and/or a partial order?

35. Suppose that R_i is a partial order on X_i, $i = 1, 2$. Show that R is a partial order on $X_1 \times X_2$ if we define

$$(x_1, x_2) \, R \, (x_1', x_2') \qquad \text{if } x_1 \, R_1 \, x_1' \text{ and } x_2 \, R_2 \, x_2'.$$

36. Let R_1 and R_2 be the relations on $\{1, 2, 3, 4\}$ given by

$$R_1 = \{(1, 1), (1, 2), (3, 4), (4, 2)\}$$
$$R_2 = \{(1, 1), (2, 1), (3, 1), (4, 4), (2, 2)\}.$$

List the elements of $R_1 \circ R_2$ and $R_2 \circ R_1$.

Give examples of relations on $\{1, 2, 3, 4\}$ having the properties specified in Exercises 37–41.

37. Reflexive, symmetric, and not transitive

38. Reflexive, not symmetric, and not transitive

39. Reflexive, antisymmetric, and not transitive

40. Not reflexive, symmetric, not antisymmetric, and transitive
41. Not reflexive, not symmetric, and transitive

Let R and S be relations on X. Determine whether each statement in Exercises 42–54 is true or false. If the statement is true, prove it; otherwise, give a counterexample.

42. If R is transitive, then R^{-1} is transitive.
43. If R and S are reflexive, then $R \cup S$ is reflexive.
44. If R and S are reflexive, then $R \cap S$ is reflexive.
45. If R and S are reflexive, then $R \circ S$ is reflexive.
46. If R is reflexive, then R^{-1} is reflexive.
47. If R and S are symmetric, then $R \cup S$ is symmetric.
48. If R and S are symmetric, then $R \cap S$ is symmetric.
49. If R and S are symmetric, then $R \circ S$ is symmetric.
50. If R is symmetric, then R^{-1} is symmetric.
51. If R and S are antisymmetric, then $R \cup S$ is antisymmetric.
52. If R and S are antisymmetric, then $R \cap S$ is antisymmetric.
53. If R and S are antisymmetric, then $R \circ S$ is antisymmetric.
54. If R is antisymmetric, then R^{-1} is antisymmetric.
55. How many relations are there on an n-element set?

In Exercises 56–58, determine whether each relation R defined on the collection of all nonempty subsets of real numbers is reflexive, symmetric, antisymmetric, transitive, and/or a partial order.

56. $(A, B) \in R$ if for every $\varepsilon > 0$, there exists $a \in A$ and $b \in B$ with $|a - b| < \varepsilon$.
57. $(A, B) \in R$ if for every $a \in A$ and $\varepsilon > 0$, there exists $b \in B$ with $|a - b| < \varepsilon$.
58. $(A, B) \in R$ if for every $a \in A$, $b \in B$, and $\varepsilon > 0$, there exists $a' \in A$ and $b' \in B$ with $|a - b'| < \varepsilon$ and $|a' - b| < \varepsilon$.
59. What is wrong with the following argument, which supposedly shows that any relation R on X that is symmetric and transitive is reflexive?

    Let $x \in X$. Using symmetry, we have (x, y) and (y, x) both in R. Since $(x, y), (y, x) \in R$, by transitivity we have $(x, x) \in R$. Therefore, R is reflexive.

3.4 ➔ Equivalence Relations

Suppose that we have a set X of 10 balls, each of which is either red, blue, or green (see Figure 3.4.1). If we divide the balls into sets R, B, and G according to color, the family $\{R, B, G\}$ is a partition of X. (Recall that in Section 1.1, we defined a partition of a set X to be a collection $\mathcal{S}$ of nonempty subsets of X such that every element in X belongs to exactly one member of $\mathcal{S}$.)

A partition can be used to define a relation. If $\mathcal{S}$ is a partition of X, we may define $x \, R \, y$ to mean that for some set $S \in \mathcal{S}$, both x and y belong to S. For the example of Figure 3.4.1, the relation obtained could be described as "is the same color as." The next theorem shows that such a relation is always reflexive, symmetric, and transitive.

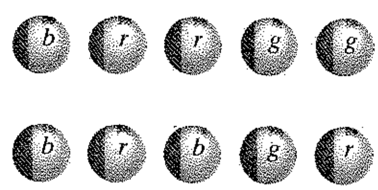


Figure 3.4.1 A set of colored balls.

Theorem 3.4.1 *Let $\mathcal{S}$ be a partition of a set X. Define $x \, R \, y$ to mean that for some set S in $\mathcal{S}$, both x and y belong to S. Then R is reflexive, symmetric, and transitive.*

Proof Let $x \in X$. By the definition of partition, x belongs to some member S of $\mathcal{S}$. Thus $x \, R \, x$ and R is reflexive.

Suppose that $x \, R \, y$. Then both x and y belong to some set $S \in \mathcal{S}$. Since both y and x belong to S, $y \, R \, x$ and R is symmetric.

Finally, suppose that $x \, R \, y$ and $y \, R \, z$. Then both x and y belong to some set $S \in \mathcal{S}$ and both y and z belong to some set $T \in \mathcal{S}$. Since y belongs to exactly one member of $\mathcal{S}$, we must have $S = T$. Therefore, both x and z belong to S and $x \, R \, z$. We have shown that R is transitive.

In the digraph of an equivalence relation, an equivalence class is a largest subgraph of the original digraph having the property that for any vertices v and w in G, there is a directed edge from v to w.

A partition of a set gives rise to an equivalence relation. If $X_1, \ldots, X_n$ is a partition of a set X and we define $x\, R\, y$ if for some i, x and y both belong to X_i, then R is an equivalence relation on X. The equivalence classes turn out to be $X_1, \ldots, X_n$. Thus, "equivalence relation" and "partition of a set" are different views of the same situation. An equivalence relation on X gives rise to a partition of X (namely, the equivalence classes), and a partition of X gives rise to an equivalence relation (namely, x is related to y if x and y are in the same set in the partition). This latter fact can be used to solve certain problems. If you are asked to find an equivalence relation, you can either find the equivalence relation directly or construct a partition and then use the associated equivalence relation. Similarly, if you are asked to find a partition, you can either find the partition directly or construct an equivalence relation and then take the equivalence classes as your partition.

Section Review Exercises

1. Define *equivalence relation.* Give an example of an equivalence relation. Give an example of a relation that is *not* an equivalence relation.
2. Define *equivalence class.* How do we denote an equivalence class? Give an example of an equivalence class for your equivalence relation of Exercise 1.
3. Explain the relationship between a partition of a set and an equivalence relation.

Exercises

In Exercises 1–8, determine whether the given relation is an equivalence relation on $\{1, 2, 3, 4, 5\}$. *If the relation is an equivalence relation, list the equivalence classes. (In Exercises 5–8,* $x, y \in \{1, 2, 3, 4, 5\}$.*)*

1. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1)\}$
2. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (3, 4), (4, 3)\}$
3. $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$
4. $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 5), (5, 1), (3, 5), (5, 3), (1, 3), (3, 1)\}$
5. $\{(x, y) \mid 1 \le x \le 5 \text{ and } 1 \le y \le 5\}$
6. $\{(x, y) \mid 4 \text{ divides } x - y\}$
7. $\{(x, y) \mid 3 \text{ divides } x + y\}$
8. $\{(x, y) \mid x \text{ divides } 2 - y\}$

In Exercises 9–14, determine whether the given relation is an equivalence relation on the set of all people.

9. $\{(x, y) \mid x$ and y are the same height$\}$
10. $\{(x, y) \mid x$ and y have, at some time, lived in the same country$\}$
11. $\{(x, y) \mid x$ and y have the same first name$\}$
12. $\{(x, y) \mid x$ is taller than $y\}$
13. $\{(x, y) \mid x$ and y have the same parents$\}$
14. $\{(x, y) \mid x$ and y have the same color hair$\}$

In Exercises 15–20, list the members of the equivalence relation on $\{1, 2, 3, 4\}$ *defined (as in Theorem 3.4.1) by the given partition. Also, find the equivalence classes* $[1], [2], [3]$, *and* $[4]$.

15. $\{\{1, 2\}, \{3, 4\}\}$
16. $\{\{1\}, \{2\}, \{3, 4\}\}$
17. $\{\{1\}, \{2\}, \{3\}, \{4\}\}$
18. $\{\{1, 2, 3\}, \{4\}\}$
19. $\{\{1, 2, 3, 4\}\}$
20. $\{\{1\}, \{2, 4\}, \{3\}\}$

In Exercises 21–23, let $X = \{1, 2, 3, 4, 5\}$, $Y = \{3, 4\}$, *and* $C = \{1, 3\}$. *Define the relation* R *on* $\mathcal{P}(X)$, *the set of all subsets of* X, *as*

$$A\, R\, B \quad \text{if } A \cup Y = B \cup Y.$$

21. Show that R is an equivalence relation.
22. List the elements of $[C]$, the equivalence class containing C.
23. How many distinct equivalence classes are there?
24. Let

    X = {San Francisco, Pittsburgh, Chicago, San Diego, Philadelphia, Los Angeles}.

    Define a relation R on X as $x\, R\, y$ if x and y are in the same state.

    (a) Show that R is an equivalence relation.

    (b) List the equivalence classes of X.
25. If an equivalence relation has only one equivalence class, what must the relation look like?
26. If R is an equivalence relation on a finite set X and $|X| = |R|$, what must the relation look like?

27. By listing ordered pairs, give an example of an equivalence relation on $\{1, 2, 3, 4, 5, 6\}$ having exactly four equivalence classes.

28. How many equivalence relations are there on the set $\{1, 2, 3\}$?

29. Let R be a reflexive relation on X satisfying: for all $x, y, z \in X$, if $x\,R\,y$ and $y\,R\,z$, then $z\,R\,x$. Prove that R is an equivalence relation.

30. Define a relation R on $\mathbf{R}^{\mathbf{R}}$, the set of functions from $\mathbf{R}$ to $\mathbf{R}$, by $f\,R\,g$ if $f(0) = g(0)$. Prove that R is an equivalence relation on $\mathbf{R}^{\mathbf{R}}$. Let $f(x) = x$ for all $x \in \mathbf{R}$. Describe $[f]$.

31. Let $X = \{1, 2, \ldots, 10\}$. Define a relation R on $X \times X$ by $(a, b)\,R\,(c, d)$ if $a + d = b + c$.
    (a) Show that R is an equivalence relation on $X \times X$.
    (b) List one member of each equivalence class of $X \times X$.

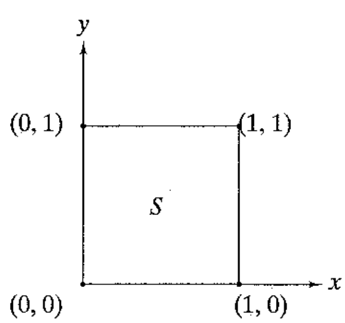
32. Let $X = \{1, 2, \ldots, 10\}$. Define a relation R on $X \times X$ by $(a, b)\,R\,(c, d)$ if $ad = bc$.
    (a) Show that R is an equivalence relation on $X \times X$.
    (b) List one member of each equivalence class of $X \times X$.
    (c) Describe the relation R in familiar terms.

33. Let R be a reflexive and transitive relation on X. Show that $R \cap R^{-1}$ is an equivalence relation on X.

34. Let R_1 and R_2 be equivalence relations on X.
    (a) Show that $R_1 \cap R_2$ is an equivalence relation on X.
    (b) Describe the equivalence classes of $R_1 \cap R_2$ in terms of the equivalence classes of R_1 and the equivalence classes of R_2.

35. Suppose that $\mathcal{S}$ is a collection of subsets of a set X and $X = \cup\mathcal{S}$. (It is not assumed that the family $\mathcal{S}$ is pairwise disjoint.) Define $x\,R\,y$ to mean that for some set $S \in \mathcal{S}$, both x and y are in S. Is R necessarily reflexive, symmetric, or transitive?

36. Let S be a unit square including the interior, as shown in the following figure.



Define a relation R on S by $(x, y)\,R\,(x', y')$ if $(x = x'$ and $y = y')$, or $(y = y'$ and $x = 0$ and $x' = 1)$, or $(y = y'$ and $x = 1$ and $x' = 0)$.
    (a) Show that R is an equivalence relation on S.
    (b) If points in the same equivalence class are glued together, how would you describe the figure formed?

37. Let S be a unit square including the interior (as in Exercise 36). Define a relation R' on S by $(x, y)\,R\,p(x', y')$ if $(x = x'$ and $y = y')$, or $(y = y'$ and $x = 0$ and $x' = 1)$, or $(y = y'$ and $x = 1$ and $x' = 0)$, or $(x = x'$ and $y = 0$ and $y' = 1)$, or $(x = x'$ and $y = 1$ and $y' = 0)$. Let

$$R = R' \cup \{((0, 0), (1, 1)), ((0, 1), (1, 0)), ((1, 0), (0, 1)), ((1, 1), (0, 0))\}.$$

    (a) Show that R is an equivalence relation on S.
    (b) If points in the same equivalence class are glued together, how would you describe the figure formed?

38. Let f be a function from X to Y. Define a relation R on X by

$$x\,R\,y \qquad \text{if } f(x) = f(y).$$

Show that R is an equivalence relation on X.

39. Let f be a characteristic function in X. ("Characteristic function" is defined before Exercise 82, Section 3.1.) Define a relation R on X by $x\,R\,y$ if $f(x) = f(y)$. According to the preceding exercise, R is an equivalence relation. What are the equivalence classes?

40. Let f be a function from X onto Y. Let

$$\mathcal{S} = \{f^{-1}(\{y\}) \mid y \in Y\}.$$

[The definition of $f^{-1}(B)$, where B is a set, precedes Exercise 70, Section 3.1.] Show that $\mathcal{S}$ is a partition of X. Describe an equivalence relation that gives rise to this partition.

41. Let R be an equivalence relation on a set A. Define a function f from A to the set of equivalence classes of A by the rule

$$f(x) = [x].$$

When do we have $f(x) = f(y)$?

42. Let R be an equivalence relation on a set A. Suppose that g is a function from A into a set X having the property that if $x\,R\,y$, then $g(x) = g(y)$. Show that

$$h([x]) = g(x)$$

defines a function from the set of equivalence classes of A into X. [What needs to be shown is that h *uniquely* assigns a value to $[x]$; that is, if $[x] = [y]$, then $g(x) = g(y)$.]

43. Suppose that a relation R on a set X is symmetric and transitive but not reflexive. Suppose, in particular, that $(b, b) \notin R$. Prove that the pseudo equivalence class $[b]$ (see Example 3.4.15) is empty.

44. Prove that if a relation R on a set X is not symmetric but transitive, the collection of pseudo equivalence classes (see Example 3.4.15) does not partition X.

45. Prove that if a relation R on a set X is reflexive but not symmetric, the collection of pseudo equivalence classes (see Example 3.4.15) does not partition X.

46. Prove that if a relation R on a set X is reflexive but not transitive, the collection of pseudo equivalence classes (see Example 3.4.15) does not partition X.

47. Give an example of a set X and a relation R on X that is not reflexive, not symmetric, and not transitive, but for which the collection of pseudo equivalence classes (see Example 3.4.15) partitions X.

Let R_1 be a relation from X to Y and let R_2 be a relation from Y to Z. Let A_1 be the matrix of R_1 and let A_2 be the matrix of R_2. The matrix of the relation $R_2 \circ R_1$ is obtained by replacing each nonzero term in the matrix product A_1A_2 by 1.

To test whether a relation is transitive, let A be its matrix. Compute A^2. The relation is transitive if and only if whenever entry i, j in A^2 is nonzero, entry i, j in A is also nonzero.

Section Review Exercises

1. What is the matrix of a relation?
2. Given the matrix of a relation, how can we determine whether the relation is reflexive?
3. Given the matrix of a relation, how can we determine whether the relation is symmetric?
4. Given the matrix of a relation, how can we determine whether the relation is transitive?
5. Given the matrix A_1 of the relation R_1 and the matrix A_2 of the relation R_2, explain how to obtain the matrix of the relation $R_2 \circ R_1$.

Exercises

In Exercises 1–3, find the matrix of the relation R from X to Y relative to the orderings given.

1. $R = \{(1, \delta), (2, \alpha), (2, \Sigma), (3, \beta), (3, \Sigma)\}$; ordering of X: 1, 2, 3; ordering of Y: $\alpha, \beta, \Sigma, \delta$
2. R as in Exercise 1; ordering of X: 3, 2, 1; ordering of Y: $\Sigma, \beta, \alpha, \delta$
3. $R = \{(x, a), (x, c), (y, a), (y, b), (z, d)\}$; ordering of X: x, y, z; ordering of Y: a, b, c, d

In Exercises 4–6, find the matrix of the relation R on X relative to the ordering given.

4. $R = \{(1, 2), (2, 3), (3, 4), (4, 5)\}$; ordering of X: 1, 2, 3, 4, 5
5. R as in Exercise 4; ordering of X: 5, 3, 1, 2, 4
6. $R = \{(x, y) \mid x < y\}$; ordering of X: 1, 2, 3, 4
7. Find matrices that represent the relations of Exercises 13–16, Section 3.3.

In Exercises 8–10, write the relation R, given by the matrix, as a set of ordered pairs.

8. $$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

9. $$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

10. $$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

11. How can we quickly determine whether a relation R is antisymmetric by examining the matrix of R (relative to some ordering)?
12. Tell whether the relation of Exercise 10 is reflexive, symmetric, transitive, antisymmetric, a partial order, and/or an equivalence relation.
13. Given the matrix of a relation R from X to Y, how can we find the matrix of the inverse relation R^{-1}?
14. Find the matrix of the inverse of each of the relations of Exercises 8 and 9.
15. Use the matrix of the relation to test for transitivity (see Examples 3.5.7 and 3.5.8) for the relations of Exercises 4, 6, and 10.

In Exercises 16–18, find

(a) The matrix A_1 of the relation R_1 (relative to the given orderings).

(b) The matrix A_2 of the relation R_2 (relative to the given orderings).

(c) The matrix product A_1A_2.

(d) Use the result of part (c) to find the matrix of the relation $R_2 \circ R_1$.

(e) Use the result of part (d) to find the relation $R_2 \circ R_1$ (as a set of ordered pairs).

16. $R_1 = \{(1, x), (1, y), (2, x), (3, x)\}$; $R_2 = \{(x, b), (y, b), (y, a), (y, c)\}$; orderings: 1, 2, 3; x, y; a, b, c
17. $R_1 = \{(x, y) \mid x \text{ divides } y\}$; R_1 is from X to Y; $R_2 = \{(y, z) \mid y > z\}$; R_2 is from Y to Z; ordering of X and Y: 2, 3, 4, 5; ordering of Z: 1, 2, 3, 4
18. $R_1 = \{(x, y) \mid x + y \le 6\}$; R_1 is from X to Y; $R_2 = \{(y, z) \mid y = z + 1\}$; R_2 is from Y to Z; ordering of X, Y, and Z: 1, 2, 3, 4, 5