MAE 384. Advanced Mathematical Methods for Engineers.
Homework Assignment 6. Due March 29.

1. Consider the differential equation:

$$\frac{dy}{dt} = -\frac{1}{2} y \sin^2 t$$

with initial condition given by

$$y(0) = 1$$

Solve this equation from $t = 0$ to $t = 10\pi$ using the following methods:

(a) Solve analytically by separating variables and integrating.

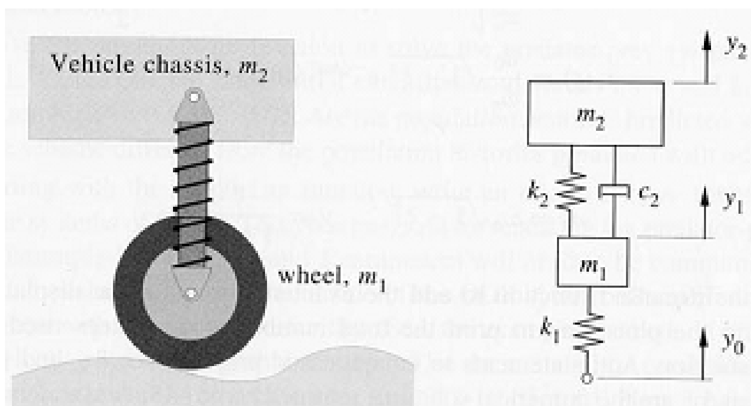
(b) Solve using the Euler implicit method. Use a time step size of 0.5.

(c) Solve using the 4th-order Runge-Kutta method. Once again, use a time step size of 0.5.

(d) Solve using the MATLAB function ode45.

(e) Compare the results for the four methods.

2. A car and its suspension system traveling over a bumpy road can be modeled as a mass/spring/damper system. In this model, $y_1$is the vertical motion of the wheel center of mass, y_2 is the vertical motion of the car chassis, and y_0 represents the displacement of the bottom of the tire due to the variation in the road surface.



Spring/mass/damper model for an automobile suspension system.

Applying Newton's law to the two masses yields a system of second-order equations:

$$\begin{aligned} m_1\ddot{y}_1 + c_2(\dot{y}_1 - \dot{y}_2) + k_2(y_1 - y_2) + k_1 y_1 &= k_1 y_0 \\ m_2\ddot{y}_2 - c_2(\dot{y}_1 - \dot{y}_2) - k_2(y_1 - y_2) &= 0 \end{aligned}$$

(a) Convert the two second-order ODE's into a system of 4 first-order ODE's. Write them in standard form.

(b) Assume the car hits a sharp bump in the road at $t = 0$ so that

$$y_0(t) = \begin{cases} 0.2t & 0 \le t < 1 \text{ s} \\ 0.4 - 0.2t & 1 < t < 2 \text{ s} \\ 0 & t > 2 \text{ s} \end{cases}$$

Create a MATLAB function that returns the right hand sides of the equations derived in part (a) for an input t and input values of the displacements and velocities.

(c) Solve the system on the time interval [0 30] seconds using the MATLAB function ode45. Find the displacement and velocity of the chassis and the wheel as a function of time. Use the following data: $m_1 = 70$ kg, $m_2 = 1900$ kg, $k_1 = 5000$ N/m, $k_2 = 500$ N/m, $c_2 = 600$ N-s/m. Does this seem like a good design?

Present your results for the above problems in an appropriate fashion. For problem 1, be sure to *include a comparison of the numerical methods* with each other and with the true solution. Be sure to discuss your findings with respect to the notions of stability and accuracy of the numerical methods. For problem 2, ensure that your results are easily interpreted by a reader. Students receiving a score of 70% or above on these two problems will receive credit for Outcome #6.

3. Consider the 4^{th}-order Runge-Kutta method as derived in class:

$$\begin{aligned}
y^*_{n+1/2} &= y_n + \frac{\Delta t}{2}\, f(t_n, y_n) \\
y^{**}_{n+1/2} &= y_n + \frac{\Delta t}{2}\, f(t_{n+1/2}, y^*_{n+1/2}) \\
y^*_{n+1} &= y_n + \Delta t\, f(t_{n+1/2}, y^{**}_{n+1/2}) \\
y_{n+1} &= y_n + \frac{\Delta t}{6}\left[f(t_n, y_n) + 2f(t_{n+1/2}, y^*_{n+1/2}) + 2f(t_{n+1/2}, y^{**}_{n+1/2}) + f(t_{n+1}, y^*_{n+1})\right]
\end{aligned}$$

   Find the stability limit for the simple ODE

$$\frac{dy}{dt} = -\alpha y,\; y(0) = y_0$$

   That is, find the limit on $\alpha\Delta t$ in order that the method remains stable. Is this method conditionally stable, unconditionally stable, or unconditionally unstable? Comment on its stability relative to the Euler explicit method. Be sure to show your work.